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Mathematical Methods of Quantum Mechanics
4th Exercise Sheet

8. Let H be a semibounded symmetric operator in the Hilbert space $\mathcal{H}$, i.e. there exists a $c \in \mathbb{R}$ such that $\langle Hu, u\rangle \geq c\langle u, u\rangle$. Prove that the following are equivalent:

(i) $H^* = H$.
(ii) H is closed and for some $\lambda > -c$ one has ker$(H^* + \lambda) = \{0\}$.
(iii) ran$(H + \lambda) = \mathcal{H}$.

Show moreover that the following statements are equivalent:

(i) H is essentially self-adjoint.
(ii) For some $\lambda > -c$ one has ker$(H^* + \lambda) = \{0\}$.
(iii) ran$(H + \lambda)$ is dense in $\mathcal{H}$.

9. Let H be a semibounded self-adjoint operator, and A be a symmetric operator with $D(H) \subset D(A)$. Show the following: if

$$(*) \quad \|A\phi\| \leq a\|H\phi\| + b\|\phi\| \quad (\phi \in D(H))$$

for some $0 \leq a < 1, b \geq 0$. Then $H + A$ is self-adjoint on $D(H)$.
Hint: Let $a_0 := \inf\{a : (*) \text{ holds for some } b\}$ and show

$$\lim_{\lambda\to\infty} \|A(H + \lambda)^{-1}\| = a_0.$$