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Mathematical Methods of Quantum Mechanics
5th Exercise Sheet

10. Let H be a symmetric, semibounded operator in the Hilbert space $\mathcal{H}$ (with domain $D(H)$), and A an operator with $D(H) \subseteq D(A)$. Suppose that there exist $\tilde{a}, \tilde{b}$ such that

$$(**) \quad \|A\phi\|^2 \leq \tilde{a}^2 \|H\phi\|^2 + \tilde{b}^2 \|\phi\|^2 \quad (\phi \in D(H)).$$

Show that then the condition (*) from exercise 9 holds:

$$(*) \quad \|A\phi\| \leq a\|H\phi\| + b\|\phi\| \quad (\phi \in D(H))$$

and that

$$\inf\{a : (*) \text{ holds for some } b > 0\} = \inf\{\tilde{a} : (**) \text{ holds for some } \tilde{b} > 0\}.$$

11. (*Magnetic Schrödinger Hamiltonian.*) The magnetic Schrödinger Hamiltonian is formally given by

$$H\phi = (-i\nabla - e\mathbf{A})^2\phi + V\phi$$

with electric charge e and magnetic vector potential $\mathbf{A}$. Show that in case of $V \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ and $\mathbf{A} \in L^4(\mathbb{R}^3) + L^\infty(\mathbb{R}^3), \nabla \cdot \mathbf{A} \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, the Hamiltonian is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$ when written in the form

$$H\phi = -\Delta\phi - 2ie\mathbf{A} \cdot \nabla\phi - ie(\nabla \cdot \mathbf{A})\phi + e^2(\mathbf{A} \cdot \mathbf{A})\phi + V\phi.$$

Hint: If the exercise seems too difficult, try to take $\mathbf{A} \in L^\infty(\mathbb{R}^3), \nabla \cdot \mathbf{A} \in L^\infty(\mathbb{R}^3)$.