Exercise 37: Standing waves

Standing waves are solutions of the $n$-dimensional wave equation of the form $u(x,t) = v(x) \sin(\omega t + \varphi)$ where $\omega > 0$, $\varphi \in [0, 2\pi)$ and $v \in C^2(\mathbb{R}^n)$.

a) Find a differential equation for $v$.

b) Determine all standing waves in dimension 1.

c) Determine all standing waves in dimension 3 that are radially symmetric with respect to the space variable.

Exercise 38: Decay of 3D waves

Let $u \in C^2$ a solution of the 3D wave equation and $U(0) < \infty$ for

$$U(t) := \sum_{\alpha \leq 2} \int_{\mathbb{R}^3} |D^\alpha u(x,t)| \, dx.$$ 

a) Show that there is a constant $K > 0$ independent of $u$ such that $|u(x,t)| \leq \frac{K}{t} U(0)$ for all $t \geq 1$.

b) Applying this result to $v(x,t) = u(x, T - t)$ for some large enough $T > 0$ prove that

$$\lim_{t \to \infty} \frac{U(t)}{t} = 0 \implies u \equiv 0$$

*Hint:* In a) write the solution in the form $u(x,t) = \int_{\partial B_t(x)} f \nu \, dS$ and apply divergence theorem.
Exercise 39: An explicit 3D wave

Solve the following 3D initial value problem

\[ u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_{>0} \]
\[ u(x, 0) = 0 \quad \text{on } \mathbb{R}^3 \]
\[ u_t(x, 0) = x_1^2 + x_1 x_2 + x_3^2 \quad \text{on } \mathbb{R}^3 \]

Exercise 40: Elastic waves

Consider the 3D elastic wave equation

\[ \left( \frac{\partial^2}{\partial t^2} - c_1^2 \Delta \right) \left( \frac{\partial^2}{\partial t^2} - c_2^2 \Delta \right) u = 0 \]

a) Show that the spherical mean \( M_u \) satisfies the differential equation

\[ \left( \frac{\partial^2}{\partial t^2} - c_1^2 \frac{\partial^2}{\partial r^2} \right) \left( \frac{\partial^2}{\partial t^2} - c_2^2 \frac{\partial^2}{\partial r^2} \right) v = 0 \]

where \( v(r, t) = r M_u(r, t) \).

b) Prove that there are functions \( F_1, F_2, G_1, G_2 \) in \( C^2(\mathbb{R}) \) such that

\[ v(r, t) = F_1(r + c_1 t) + F_2(r - c_1 t) + G_1(r + c_2 t) + G_2(r - c_2 t) \]

c) Solve the general initial value problem of the 3D elastic wave equation where \( v(r, 0), v_t(r, 0), v_{tt}(r, 0), v_{ttt}(r, 0) \) are given functions that are sufficiently smooth.

*Hint:* In b) define a suitable function \( U : \mathbb{R}^4 \rightarrow \mathbb{R} \) such that \( U(z) = u(r, t) \).

In c) it may be convenient to solve the system for \( v(r, 0) = v_t(r, 0) = 0 \) first. (Your linear algebra skills are required!)