Exercise 49: Solving the Cauchy problem with complete integrals

In this exercise we will discuss how to solve the Cauchy problem

\[ u_x u_y = 1 \quad \text{and} \quad u(0, y) = \log(y) \quad (y > 0) \]

using the complete integral \( u(x, y; a, b) \) calculated in exercise 47.

a) Determine functions \( a(s), b(s) \) such that \( (x, y) \mapsto u(x, y; a(s), b(s)) \) solves the initial conditions and the strip condition

\[ \frac{d}{ds} \log(s) = Du(0, s, a(s), b(s)) \cdot (0, 1) \]

b) Calculate the envelope \( u^*(x, y) \) of this family of solutions.

c) Check that \( u^* \) is a solution of the above problem.

Exercise 50: First Order PDE V, Nonuniqueness

Consider the problem

\[ u_x^2 + u_y^2 = 1 \quad \text{and} \quad u(\cos(\varphi), \sin(\varphi)) = 0 \]

Apply the method of characteristics to determine two solutions of the problem. 

*Hint:* There are two possible choices for the initial data \( P(0) \).

Exercise 51: First Order PDE VI

Solve the problem

\[ u_x^3 - u_y = 0 \quad \text{and} \quad u(x, 0) = 2x^{3/2} \quad (x > 0) \]
Exercise 52: Picone’s example

Let \( u \in C^1(B_1(0)) \) a solution of

\[
a(x, y)u_x + b(x, y)u_y = -u \quad \text{in } B_1(0)
\]

and \( a(x, y)x + b(x, y)y > 0 \) on \( \partial B_1(0) \). In the following we write \( u(r, \varphi) \) using polar coordinates.

a) Prove that if \( u \) attains its maximum in \( x_0 \in \partial B_1(0) \) then \( \frac{\partial u}{\partial r}(x_0) \geq 0 \) and \( \frac{\partial u}{\partial \varphi}(x_0) = 0 \). What (in)equalities do we obtain for a minimum?

b) Use a) to show \( \max_{B_1(0)} u \leq 0 \) as well as \( \min_{B_1(0)} u \geq 0 \), i.e. \( u \equiv 0 \).

c) Solve the above problem explicitly for \( a(x, y) = x, b(x, y) = cy \) with initial conditions \( u(1, \varphi) = f(\varphi) \) for \( c = 0 \) and \( c = 1 \) for a given function \( f \in C^1([0, 2\pi]) \).