Exercise 25: The heat equation and convexity

a) Let \( f : \mathbb{R} \to \mathbb{R} \) a convex function satisfying the growth condition \( f(x) \leq Me^{ax^2} \) for given \( M, a > 0 \). Show that the function

\[
u(x, t) = \int_{\mathbb{R}} \Phi(x-y, t)f(y) \, dy \tag{1}
\]

is well-defined for sufficiently small \( 0 < t < t_0 \), solves the heat equation and is a convex function in \( x \) for all \( 0 < t < t_0 \).

b) Let \( f \geq 0 \). Show that \( t_2 \geq t_1 > 0 \) implies \( u(x, t_2) \geq u(x, t_1) \) for \( u \) given by (1).

Exercise 26: Three Symmetry groups of the heat equation

Show that for any solution \( u(x, t) \) of the heat equation the function \( u_\varepsilon(x, t) \) defined by

a) \( u_\varepsilon(x, t) = u(x, t + \varepsilon) \)

b) \( u_\varepsilon(x, t) = e^{-\varepsilon x + \varepsilon^2 t} u(x - 2\varepsilon t, t) \)

c) \( u_\varepsilon(x, t) = \frac{1}{\sqrt{1 + 4\varepsilon t}} \exp \left( \frac{-\varepsilon x^2}{1 + 4\varepsilon t} \right) u \left( \frac{x}{1 + 4\varepsilon t}, \frac{t}{1 + 4\varepsilon t} \right) \)

is still a solution for any given \( \varepsilon > 0 \) sufficiently small. Show that via a) and c) a constant solution can be transformed into the fundamental solution of the 1D heat equation.
Exercise 27: Weak maximum principle for a parabolic equation

We consider a function $u : U_T \rightarrow \mathbb{R}$ for a bounded domain $U$ and $U_T = U \times (0, T]$ with the property

$$u_t - \Delta u + b(x, t) \cdot Du + c(x, t)u \geq 0 \quad \text{in } U_T$$

Show that if $u \geq 0$ on $\Gamma_T := \partial U \times [0, T] \cup U \times \{0\}$ and $c \geq 0$ in $U_T$, then $u \geq 0$ on $U_T$. Conclude that in the case $c \equiv 0$ we have

$$\min_{U_T} u = \min_{\Gamma_T} u$$

Exercise 28: A special solution of the 1D heat equation

Determine a bounded solution of the following problem

$$u_t - u_{xx} = 0 \quad \text{in } \mathbb{R} \times (0, \infty)$$

$$u(x, 0) = 1_{(0, \infty)}(x) \quad \text{on } \mathbb{R} \times \{0\}$$