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Partial Differential Equations

Exercise Sheet 3

Exercise 1

a) Suppose $u \in C^0(\overline{\Omega})$ satisfies the mean value property in Ω and $B_R(x_0) \subset \Omega$. Show that for every multi-index α with $|\alpha| = m$

$$|D^\alpha u(x_0)| \leq \frac{n^m e^{m-1} m!}{R^m} \max_{\overline{B_R(x_0)}} |u|.$$

b) Let $u : \mathbb{R}^n \to \mathbb{R}$ be harmonic and $\beta \geq 0$. Suppose

$$\limsup_{|x|\to\infty} \frac{|u(x)|}{|x|^\beta} < \infty.$$

Show that u is a polynomial of a degree $\leq \beta$.

Exercise 2

Suppose $u \in C^2(\overline{B_1^+})$ is harmonic in $B_1^+ = \{x \in B_1 : x_n > 0\}$ with $u = 0$ on $\{x_n = 0\} \cap B_1$. Prove that the odd extension of u (i.e. $u(x_1, \ldots, x_{n-1}, x_n) := -u(x_1, \ldots, x_{n-1}, -x_n)$ for $x_n < 0$) to B_1 is harmonic in B_1.

Exercise 3

Let $u \in C^2(\mathbb{R}^n)$ be a nonzero harmonic function and let $u_r = \frac{\partial u}{\partial r}$ denote its radial derivative. Given $r > 0$, define scale invariant quantities $I(r)$, $E(r)$, and the frequency function $U(r)$ by

$$\begin{aligned} I(r) &= r^{1-n} \int_{\partial B_r} u^2 \, d\mu_{\partial B_r} \\ E(r) &= r^{2-n} \int_{B_r} |Du|^2 \, d\mu = r^{2-n} \int_{\partial B_r} u u_r \, d\mu_{\partial B_r} \\ U(r) &= \frac{E(r)}{I(r)}. \end{aligned}$$

Prove the following:

a) $I'(r) = 2\,r^{1-n} \int_{\partial B_r} u u_r \, d\mu_{\partial B_r} = \frac{2E(r)}{r}$.

b) $E'(r) = 2\,r^{2-n} \int_{\partial B_r} |u_r|^2 \, d\mu_{\partial B_r}$.

c) $\frac{d}{dr} \log U(r) = \frac{2r^{-n}}{E(r)} \int_{\partial B_r} (r u_r - U(r) u)^2 \, d\mu_{\partial B_r} \geq 0$.

Deduce that $U(r)$ is a nondecreasing function in $r > 0$.