Karlsruhe Institute for Technology (KIT)
Institute for Analysis
Prof. Dr. Tobias Lamm
Dr. Patrick Breuning

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Partial Differential Equations

Exercise Sheet 4

Exercise 1

Let Ω be a bounded, open subset of $\mathbb{R}^n$. Prove that there exists a constant C, depending only on Ω, such that

$$\max_{\overline{\Omega}} |u| \le C(\max_{\partial\Omega} |g| + \max_{\overline{\Omega}} |f|)$$

whenever u is a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \quad u = g & \text{on } \partial\Omega. \end{cases}$$

(Hint: $-\Delta\left(u + \frac{|x|^2}{2n}\lambda\right) \le 0$, for $\lambda := \max_{\overline{\Omega}} |f|$.)

Exercise 2

Let Ω be a bounded, open subset of $\mathbb{R}^n$. Let $u_n \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be harmonic in Ω with $u_n = g_n$ on $\partial\Omega$, where $g_n \in C^0(\partial\Omega)$ with

$$\sup_{\partial\Omega} |g_n - g_m| \to 0 \qquad \text{as } n, m \to \infty.$$

Show that $(u_n)_{n\in\mathbb{N}}$ converges uniformly to a function $u : \overline{\Omega} \to \mathbb{R}$, which is harmonic in Ω and continuous on $\overline{\Omega}$.

Exercise 3

Let $\Omega \subset \mathbb{R}^n$ be bounded with C^1-boundary. For $v \in C^1(\overline{\Omega})$ the *Dirichlet energy* is defined as

$$E(v) := \frac{1}{2}\int_\Omega |Dv|^2 \, d\mu.$$

Show that for $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ the following are equivalent:

i) $\Delta u = 0$

ii) $E(u) = \min\left\{E(v) : v \in C^1(\overline{\Omega}),\ v = u \text{ on } \partial\Omega\right\}$