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Partial Differential Equations

Exercise Sheet 5

Exercise 1

Let $u : \mathbb{R}^n \to \mathbb{R}$ be harmonic, and let $F \subset \mathbb{R}^{n+1}$ be the graph of u, i.e.

$$F := \{(x, u(x)) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1} : x \in \mathbb{R}^n\}.$$

For $p \in F$ let T_pF be the tangent space of F in p, i.e. $T_pF = \operatorname{span}((e_1, \partial_1 u(y)), \ldots, (e_n, \partial_n u(y)))$, where $(y, u(y)) = p$. Show that

$$((p + T_pF) \cap F) \setminus \{p\} \neq \emptyset \quad \text{for all } p \in F.$$

Exercise 2

Let $\Omega \subset \mathbb{R}^n$ be open and let $u \in C^4(\Omega)$ be biharmonic, i.e. $\Delta(\Delta u) = 0$.

a) Show that u satisfies the following mean value property:

$$u(x) + \frac{r^2}{2n}\Delta u(x) = \fint_{\partial B_r(x)} u \, d\mu_{\partial B_r(x)} \quad \text{for all } x \in \Omega \text{ and all } \overline{B_r(x)} \subset \Omega.$$

b) Show that

$$3u(x) = 4 \fint_{\partial B_1(0)} u\left(\frac{r}{2}y + x\right) d\mu_{\partial B_1(0)}(y) - \fint_{\partial B_1(0)} u(ry + x) \, d\mu_{\partial B_1(0)}(y)$$

for all $x \in \Omega$ and all $r > 0$ with $\overline{B_r(x)} \subset \Omega$.

c) Now let u be biharmonic on all of $\mathbb{R}^n$ and let u be bounded from above and below. Show that u is constant.