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WS 2013/2014
05.12.2013

Partial Differential Equations

Exercise Sheet 7

Exercise 1

Let $f \in C_c^0(\mathbb{R}^n) = \{h \in C^0(\mathbb{R}^n) : \operatorname{spt}(h) \text{ is compact}\}$, $\Phi(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ and $u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) f(y)\, dy$ for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$.

Prove the following:

i) $\displaystyle \sup_{x\in\mathbb{R}^n} |u(x,t)| \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} |f(y)|\, dy, \quad \forall\, t > 0.$

ii) $\displaystyle \sup_{x\in\mathbb{R}^n} |D_x u(x,t)| \leq \frac{C}{t^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} |f(y)|\, dy, \quad \forall\, t > 0.$

iii) $\displaystyle \int_{\mathbb{R}^n} u(x,t)\, dx = \int_{\mathbb{R}^n} f(x)\, dx, \quad \forall\, t > 0.$

Exercise 2

Let $f \in C_c^0(\mathbb{R}^n)$ and Φ, u be as in Exercise 1. Show that

$$\lim_{t\to\infty} \left(t^{\frac{n}{2}} \sup_{x\in\mathbb{R}^n} |u(x,t) - m\Phi(x,t)| \right) = 0,$$

where $m = \int_{\mathbb{R}^n} f(y)\, dy$.

Exercise 3

Construct a solution $u \in C^\infty(\mathbb{R} \times \mathbb{R})$, $u \not\equiv 0$, of

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u = 0 & \text{on } \mathbb{R} \times (-\infty, 0]. \end{cases}$$

(Hint: Construct a suitable solution of

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R} \times \mathbb{R} \\ u(0,t) = a(t),\ u_x(0,t) = 0 & \text{for } t \in \mathbb{R}, \end{cases}$$

satisfying $u(x,t) = 0$ for $x \in \mathbb{R}^n$, $t \leq 0$. Make the ansatz

$$u(x,t) = \sum_{k=0}^{\infty} a_k(t) x^k.$$

Then determine the function $a(t)$, such that the power series converges and u satisfies the desired properties.)