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Partial Differential Equations

Exercise Sheet 10

Exercise 1

Denote by Ω the open square $\{x \in \mathbb{R}^2 : |x_1| < 1, |x_2| < 1\}$. Define

$$u(x) = \begin{cases} 1 - x_1 & \text{if } x_1 > 0, |x_2| < x_1 \\ 1 + x_1 & \text{if } x_1 < 0, |x_2| < -x_1 \\ 1 - x_2 & \text{if } x_2 > 0, |x_1| < x_2 \\ 1 + x_2 & \text{if } x_2 < 0, |x_1| < -x_2 \end{cases}$$

For which $1 \leq p \leq \infty$ does u belong to $W^{1,p}(\Omega)$?

Exercise 2

Let $B = B_{\frac{1}{2}}(0) \subset \mathbb{R}^2$ and $u : B \to \mathbb{R}$, $u(x) = \log\left(\log \frac{1}{|x|}\right)$. Show that u lies in $W^{1,2}(B)$, but not in $L^\infty(B)$.

Exercise 3

Let Ω be an open domain in $\mathbb{R}^n$, and let $u \in W^{1,1}_{\text{loc}}(\Omega)$. Show that also the functions $u^+ = \max(u, 0)$, $u^- = -\min(u, 0)$ and $|u|$ lie in $W^{1,1}_{\text{loc}}(\Omega)$, and determine the weak derivatives.

Exercise 4

Let Ω be an open domain in $\mathbb{R}^n$, and let $u \in W^{1,1}_{\text{loc}}(\Omega)$. Show that for $c \in \mathbb{R}$ the following holds:

$$Du(x) = 0 \quad \text{for almost all } x \in N_c = \{x \in \Omega : u(x) = c\}.$$