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Partial Differential Equations

Exercise Sheet 11

Exercise 1

a) Let $1 \leq p < r < q \leq \infty$. Show for $f \in L^p \cap L^q(\mathbb{R}^n)$ the following interpolation inequality:

$$\|f\|_{L^r} \leq \|f\|_{L^p}^{1-\alpha} \|f\|_{L^q}^{\alpha}, \qquad \text{where } \alpha = \frac{\frac{1}{p} - \frac{1}{r}}{\frac{1}{p} - \frac{1}{q}} \in (0,1).$$

b) Let $f \in L^p \cap W^{1,q}(\mathbb{R}^n)$, where $q < n$. Show that $f \in L^r(\mathbb{R}^n)$ for all $r \in [p, q^*]$, where $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{n}$, and show that for a constant $C = C(n,p,q,r) < \infty$ the following interpolation inequality holds:

$$\|f\|_{L^r} \leq C \|f\|_{L^p}^{1-\alpha} \|Df\|_{L^q}^{\alpha}, \qquad \text{where } \alpha = \frac{\frac{1}{p} - \frac{1}{r}}{\frac{1}{p} - \frac{1}{q^*}}.$$

This is the so called *Gagliardo-Nirenberg interpolation inequality.*

Exercise 2

Let $\Omega \subset \mathbb{R}^n$ be open.

a) Show by means of partial integration

$$\|Du\|_{L^p} \leq C \|u\|_{L^p}^{1/2} \|D^2 u\|_{L^p}^{1/2}$$

for $2 \leq p < \infty$ and all $u \in C_c^\infty(\Omega)$.

Hint: It holds $\int_\Omega |Du|^p \, dx = \sum_{i=1}^n \int_\Omega \partial_i u \partial_i u |Du|^{p-2} \, dx$.

b) Show

$$\|Du\|_{L^{2p}} \leq C \|u\|_{L^\infty}^{1/2} \|D^2 u\|_{L^p}^{1/2}$$

for $1 \leq p < \infty$ and all $u \in C_c^\infty(\Omega)$.

Exercise 3

Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $k \in \mathbb{N}_0$, and $0 < \alpha \leq 1$. Show that $\left(C^{k,\alpha}(\overline{\Omega}), \|\cdot\|_{C^{k,\alpha}(\Omega)}\right)$ is a Banach space.

Exercise 4

Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $\alpha, \beta \in [0,1]$ with $\alpha > \beta$, and $M < \infty$. Show that the set

$$\left\{u \in C^{0,\alpha}(\overline{\Omega}) : \|u\|_{C^{0,\alpha}(\Omega)} \leq M\right\}$$

is a compact subset of $C^{0,\beta}(\overline{\Omega})$.

Hint: Use the Theorem of Arzelà-Ascoli.