Karlsruhe Institute of Technology

Exercise sheet 4

Exercise 1 (Atomic part of a measure)

Let μ be a probability measure on $\mathbb{R}^n$ (with the Borel sigma algebra!). We call a measure *atom free* if for all $x \in \mathbb{R}^n$ we have

$$\lim_{r \searrow 0} \mu(B_r(x)) = 0.$$

Show that for every measure μ there is an countable set of points $(x_j) \subset \mathbb{R}^n$, $j \in J$ and numbers $0 < \mu_j < \mu(\mathbb{R}^n)$, such that $\mu_0 - \sum_{j \in J} \mu_j \delta_{x_j}$ is free of atoms using the follwing steps:

1. Show that the set $J := \{x \in \mathbb{R}^n : \lim_{r \to 0} \mu(B_r(x)) > 0\}$ is at most countable.
2. Let $\tilde{\mu}_{x_0} := \lim_{r \to 0} \mu(B_r(x_0))$. Show that

$$\mu_0 := \mu - \sum_{x_0 \in J} \tilde{\mu}_{x_0} \delta_{x = x_0}$$

is a positiv measure and free of atoms.

Exercise 2 (Best Sobolev constant on compact domains I)

For $\Omega \subset \mathbb{R}^n, p \in [1, \infty]$ we define best Sobolev constant by

$$S(p, \Omega) := \inf_{u \in C_c^\infty(\Omega, \mathbb{R}), u \not\equiv 0} \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^p}^2}.$$

1. Show that for bounded domains $S(p, \Omega) > 0$ if $p \leq 2^*$.
2. Show that $S(p, \Omega) = 0$ if $p > 2^*$ and *any* domains.
3. Show that $S(2^*, \Omega)$, $2^* = \frac{2}{n}/(n-2)$ does not depend on the domain.

Exercise 3 (Best Sobolev constant on compact domains II)

Let $\Omega \subset\subset \mathbb{R}^n$ be a bounded domain with C^1 boundary. We have seen in the lecture that for $\Omega \subset\subset \mathbb{R}^n$ there is no $u \in W^{1,2}(\Omega)$ satisfying

$$S\|u\|_{L^{2^*}(\Omega)}^2 = \|\nabla u\|_{L^2(\Omega)}^2.$$

In this exercise we will analyse minimizing sequences $(u_k)_k \in M$ of the Dirichlet energy

$$\mathcal{F}(u) := \int_\Omega |\nabla u|^2 dx$$

on $M := \{u \in W^{1,2}(\Omega) : \|u\|_{L^{2^*}(\Omega)=1}\}$. After going to a subsequence if necessary, we can furthermore assume that

$$u_k \rightharpoonup u \text{ weakly in } W^{1,2}(\Omega)$$
$$\mu_k := |\nabla u_k|^2 dx \rightharpoonup \mu \text{ weak}*$$
$$\nu_k := |u_k|^{2^*} \rightharpoonup \nu \text{ weak} * .$$

1. Use the concentration compactness lemma, to show that either

   $$\|u\|_{L^{2^*}} = 1 \qquad \text{and} \qquad \|\nabla u\|_{L^2} = S$$

   or for a suitable $x_0 \in \Omega$ we have

   $$\nu = \delta_{\{x=x_0\}} \qquad \text{and} \qquad \mu = S\delta_{\{x=x_0\}}.$$

   (Hint: Follow the lines of the proof of Theorem 5.6!)
2. Show that the first case leads to a contradiction! (Hint: Use (without proof) that weak convergence in L^p plus convergence of the L^p-norm implies strong convergence!)