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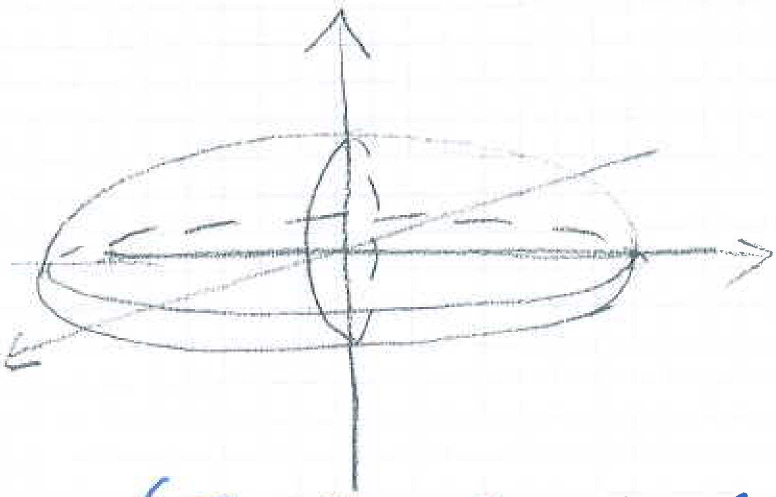
§ 6 Min-max techniques

§ 6.1 A finite dimensional example

(Lusternik & Schnirelmann, 1929)

Thm 2.1.1: On any smooth closed surface of genus 0 there exist at least 3 prime (free of crossings) geodesics.

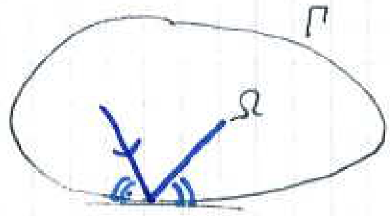


Model situation: (Shrink the 3-d thing to something flat.)

convex

We are looking for lines that come back on themselves when reflect at the boundary Γ (We are playing Billiard)

Model problem:



Ω convex with boundary Γ and we are playing "billiard" on Ω. We for lines that are coming back on themselves when reflected on the boundary Γ

Let $\gamma \in C^1(\mathbb{R} \setminus \mathbb{Z}, \mathbb{R}^2)$ be a parametrization of Γ and

$$f(s,t) = \frac{1}{2} |\gamma(s) - \gamma(t)|^2, \quad \forall s,t \in \mathbb{R}$$

$\rightsquigarrow$ $f \in C^1((\mathbb{R}/\mathbb{Z})^2)$ and $f(s,t) = f(t,s)$ and

$$\frac{\partial}{\partial s} f(s,t) = \dot{\gamma}(s) \cdot (\gamma(s) - \gamma(t))$$

$$\frac{\partial}{\partial t} f(s,t) = \dot{\gamma}(t) \cdot (\gamma(s) - \gamma(t)).$$

$\rightsquigarrow$ The line through $\gamma(s)$ and $\gamma(t)$ is reflected onto itself at $\gamma(s)$ and $\gamma(t)$ if and only if (s,t) is a critical point of f, (i.e. $df(s,t) = 0$)

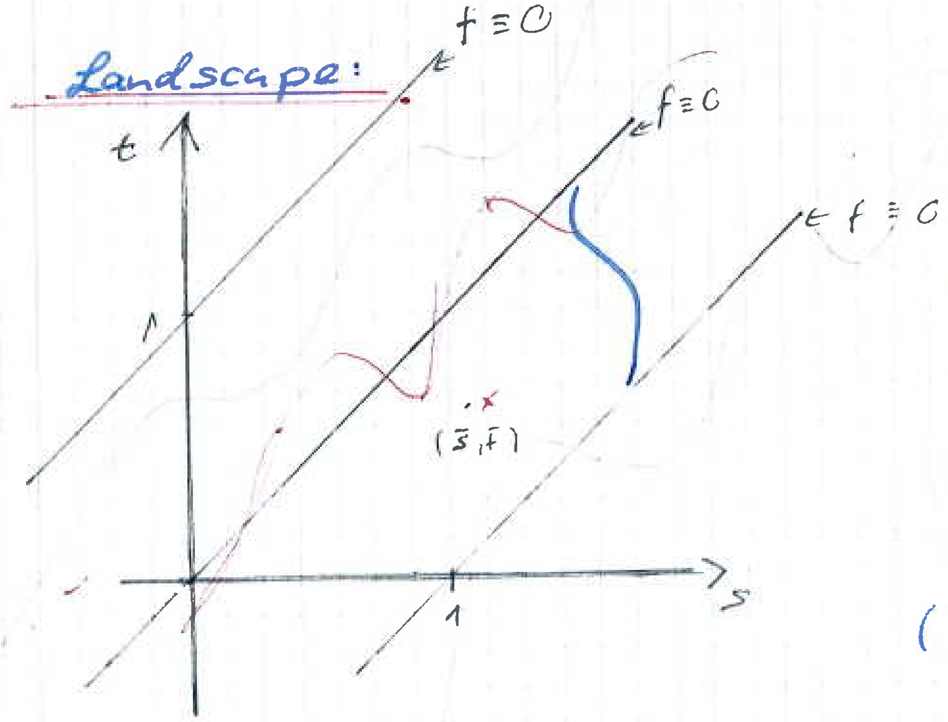
(i) Since $f(s,t) \geq 0$ and $f(s,t) = 0$ if $s = t \mod \mathbb{Z}$, we get points $s = t \mod \mathbb{Z}$ are trivial critical points (corresponding to constant curves on)

(ii) On the compact domain $T^2 = (\mathbb{R}/\mathbb{Z})^2$ f achieves its maximum at some point $(\bar{s}, \bar{t})$.

⇝ Again a critical point corresponding to the diameter of Ω.

⇝ second prime geodesic

Landscape:



We try to find a "mountain path"?

(to get a third prime geodesic

We try to find a saddle point $(\underline{s}, \underline{t})$ of f as the point of "maximal elevation" of a "path" of "least maximal height" connecting the "valleys" $\{(s,s) : s \in \mathbb{R}\}$ and $\{(s, s-1) : s \in \mathbb{R}\}$

$\rightsquigarrow$ "mountain path" ∇_0

<u>Questions:</u>

(i) Can we find a path f of least max. height?

(ii) If so, does contain a critical point $(\underline{s}, \underline{t})$ with

$$f(\underline{s}, \underline{t}) = \inf_{p \text{ path}} \sup_{(s,t) \in p} f(s,t) =: \beta_1 > 0.$$

(iii) What happens if $\beta_1 = \beta_0$?

$\boxed{\text{Ad (i):}}$ What is a path ?

Let $p_k \in C^0([0,1])$ with $p_k(0) = (s_0, s_0)$

$p_k(1) = (s_1, s_1 - 1)$ and

$$\sup_{0<r<1} f(p(r)) \to \beta_1 .$$

Problem: We cannot just deduce that

$p_k \to p_\infty$ in C^0 ∇_0

Workaround:

The sets $\Pi_k = p_k([0,1]) \subset \mathbb{T}^2$

are compact, & connected with

$$\nearrow \ \overline{\Pi}_k \cap \{(s,s) : s \in \mathbb{R}\} \neq \emptyset$$

compact

$$\searrow \ \overline{\Pi}_k \cap \{(s,s-1) : s \in \mathbb{R}\} \neq \emptyset$$

$$\rightsquigarrow \quad \overline{\Pi} = \bigcap_{\ell \in \mathbb{N}} \Big(\overline{\bigcup_{k \geq \ell} \Pi_k} \Big) \neq \emptyset \text{ is compact}$$

and connected with

$$\overline{\Pi} \cap \{(s,s) : s \in \mathbb{R}\} \neq \emptyset$$

$$\overline{\Pi} \cap \{(s,s-1) : s \in \mathbb{R}\} \neq \emptyset .$$

(Π is ~~need~~ not be a curve though ∇_0)

$\boxed{\text{Ad ii:}}$ Let us assume, that $df(s,t) \neq 0$ for all $(s,t) \in \Pi$. Then we can reach a contradiction looking at

$$\phi_\tau(\Pi)$$

where ϕ_τ is the flow in direction of $-\nabla f$, i.e. a solution of

$$\begin{cases} \frac{\partial}{\partial \tau} \phi_\tau(x) = -\nabla f(x) \\ \phi_0(x) = x \end{cases}$$

That's welldefined only if we assume that $f \in C^{1,1}$

Example 2: Let $f \in C^1(\mathbb{R}^2)$ with

$$f(x,y) = e^x - y^2.$$

Then $M_0 = \{(x,y) : f(x,y) < 0\}$ has two components

$$M_0^\pm = \{(x,y) \in M_0 : \pm y > 0\}.$$

$\rightsquigarrow$ might expect a montain path, but

$$df(x,y) = (e^x, -2y) \neq 0 \quad \forall$$

§ 6.2 Pseudo-gradient flows.

Let X be a Banach space.

Def. 6.1: Let $\mathcal{F} \in C^1(X)$.

i) $u \in X$ is critical for $\mathcal{F}$ if $d\mathcal{F}(u) = 0$, otherwise u is regular

ii) $\beta \in \mathbb{R}$ is a critical value of $\mathcal{F}$ if there exists a critical point $u \in X$ with $\mathcal{F}(u) = \beta$; otherwise, β is a regular value. //

Def. 6.2 (Pseudo-gradient vector field)
Let $\mathcal{F} \in C^1(X)$ and $\tilde{X} := \{ u \in X : d\mathcal{F}(u) \neq 0 \}$

A pseudo-gradient vector field (p.g.v.f) is a locally Lipschitz map

$$\tilde{e} : \tilde{X} \to X$$

with (i) $\| \tilde{e}(u) \|_X < 1 \quad \forall u \in \tilde{X}$

(ii) $\langle d\mathcal{F}(u), \tilde{e}(u) \rangle_{X^* \times X} > \frac{1}{2} \| d\mathcal{F}(u) \|_{X^*}$

$\forall u \in \tilde{X}$

(so $\frac{1}{2} < \| \tilde{e}(u) \|_X < 1$)

Theorem 6.3 (Palais, 1966)

For any $E \in C^1(X)$ there exists a (p.g.v.f). //

Proof:

(i) Fix $u_0 \in \tilde{X}$. From the definition of $\| d\tilde{\mathcal{F}}(u)\|_{X^*}$ we get a $v_0 = v(u_0)$ with

$$\| v_0 \|_X < 1$$

and

$$\langle d\tilde{\mathcal{F}}(u_0), v_0 \rangle > \frac{1}{2} \| d\tilde{\mathcal{F}}(u_0)\|_{X^*}.$$

Since $u \to d\tilde{\mathcal{F}}(u)$ is continuous, we have

$$\langle d\tilde{\mathcal{F}}(u), v_0 \rangle > \frac{1}{2} \| d\tilde{\mathcal{F}}(u)\|_{X^*} \quad \forall u \in U_0 = U(u_0)$$

$U(u_0)$ an open neighborhood of u_0.

(ii) The family $(U(u))_{u \in \tilde{X}}$ is an open of $\tilde{X}$

some topology & $\tilde{X}$ is metric $\rightarrow$ There is a *locally* finite refinement $(U_j)_{j \in I}$ with

$$U_j \subset U(u_z) \quad \text{for some } z \in \tilde{X}$$

and $\tilde{X} \subset \bigcup_{j \in I} U_j$.

Let $(\varphi_j)_{j \in I}$ be a locally Lipschitz partition of unity subordinate to $(U_j)_{j \in I}$.

(For instance let $\psi_j(u) = \mathrm{dist}(u, X \setminus U_j) \in C^{0,1}(X)$

and
$$\varphi_j(u) = \frac{\psi_j(u)}{\underbrace{\sum_{k \in I}}_{\text{finite sum on } U_j \ \forall j} \psi_k(u)}$$
)

(iii) Set
$$\tilde{e}(u) = \sum_{j \in I} \varphi_j(u)\, v(u_j), \quad u \in \tilde{X}.$$

Then
$$\|\tilde{e}(u)\|_X \leq \max_{j:\, u \in U_j} \|v(u_j)\|_X < 1$$

and
$$\langle d\Phi(u), \tilde{e}(u)\rangle = \sum_{j \in J} \varphi_j(u) \underbrace{\langle d\Phi(u), v(u_j)\rangle}_{\geq \frac{1}{2}\|d\Phi\| \text{ if } \varphi_j \neq 0}$$
$$> \frac{1}{2} \|d\Phi(u)\|_{X^*}$$

for all $u \in \tilde{X}$. $\square$