Let $\tilde{\mathcal{F}} \in C^1(X)$. For $\beta \in \mathbb{R}$, $\rho, \delta > 0$ we set

$$\tilde{\mathcal{F}}_\beta = \{ u \in X : \tilde{\mathcal{F}}(u) < \beta \}$$

$$K_\beta = \{ u \in X : \tilde{\mathcal{F}}(u) = \beta, \ \tilde{\mathcal{F}}'(u) = 0 \}$$

$$\mathcal{N}_{\beta,\delta} = \{ u \in X : |\tilde{\mathcal{F}}(u) - \beta| < \delta, \ \| \tilde{\mathcal{F}}'(u) \|_{X'} < \delta \}$$

$$\mathcal{U}_{\beta,\rho} = \bigcup_{u \in K_\beta} B_\rho(u)$$

<u>Lemma 6.3</u>: Suppose $\tilde{\mathcal{F}}$ satisfies $(P.S.)_\beta$. Then

(i) K_β is compact

(ii) $(\mathcal{N}_{\beta,\delta})$ and $(\mathcal{U}_{\beta,\rho})$ are fundamental systems of open neighborhoods of K_β, i.e. given any neighborhood $\mathcal{N}$ of K_β $\exists\, \delta > 0, \rho > 0$:

$$\mathcal{N} \supset \mathcal{N}_{\beta,\delta}, \quad \mathcal{N} \supset \mathcal{U}_{\beta,\rho}.$$

(iii) If $K_\beta = \emptyset$, there exists $\delta > 0$ such that $\mathcal{N}_{\beta,\delta} = \emptyset$.

//

Proof:

(i) $(u_k) \subset K_\beta$ is obviously a $(P.S)_\beta$-sequence

$\leadsto$ $u_k \to u \in K_\beta$ by $(P.S)_\beta$ for a subsequence.

(ii) Assume that the statement was wrong $\overset{\nabla}{o}$, i.e. that there is a neighborhood N of K_β such that

$$N_{\beta,\delta} \setminus N \neq \emptyset \quad \text{for all } \delta > 0.$$

$\leadsto$ $\exists \; u_k \in N_{\beta,\frac{1}{k}} \setminus N$

$\leadsto$ (u_k) is $(P.S)_\beta$-seq.

$\leadsto$ For a subsequence

$$u_k \to u \in K_\beta \subset N$$

Hence, $u_k \in N$ for k large $\lightning$

If $U_{\beta,\delta} \setminus N \neq \emptyset$ for every $\delta > 0$,

let

$$u_k \in U_{\beta,\frac{1}{k}}, \quad k \in \mathbb{N},$$

and

$$v_k \in K_\beta \quad \text{with}$$

$$\|u_k - v_k\|_X \leq \frac{1}{k}$$

(ii) $\leadsto$ For a subsequence, $v_k \to u$ and thus $u_k \to u$

Hence, $u_k \in \mathcal{N}$ for k large $\lightning$

(iii) $\mathcal{N} = \emptyset$ is a neighborhood of $K_\beta = \emptyset$.

Theorem 6.10: (THE deformation lemma)

Let $\tilde{\mathcal{F}} \in C^1(X)$ sat. $(P.-S)_\beta$ for a given $\beta \in \mathbb{R}$ and let $\mathcal{N}$ be a neighborhood of K_β, $\bar{\varepsilon} > 0$. Then there exists $0 < \varepsilon < \bar{\varepsilon}$ and $\Phi \in C^1(X \times [0,1], X)$ s. th.

(i) $\Phi(\cdot, t) : X \to X$ is a homeomorph. $\forall t \in [0,1]$.

(ii) $\Phi(u,t) = u$ if $t = 0$ or $\tilde{\mathcal{F}}'(u) = 0$ or $|\tilde{\mathcal{F}}(u) - \beta| \geq \bar{\varepsilon}$.

(iii) $t \to \tilde{\mathcal{F}}(\Phi(u,t))$ is non-increasing, $\forall u \in X$

(iv) $\Phi(\tilde{\mathcal{F}}_{\beta+\varepsilon}, 1) \subset \tilde{\mathcal{F}}_{\beta-\varepsilon} \cup \mathcal{N}$

$\Phi(\tilde{\mathcal{F}}_{\beta+\varepsilon} \setminus \mathcal{N}, 1) \subset \tilde{\mathcal{F}}_{\beta-\varepsilon}$

Proof: (i) $\mathcal{N}$ is a neighborhood of K_β

$\overset{\sim}{\to}$ Lemma $\quad \exists$ $\delta, \varrho, \bar\varepsilon < 1$ such that

$$\mathcal{N} \supset U_{\beta, 2\varrho} \supset U_{\beta\varrho} \supset \mathcal{N}_{\beta,\delta} \supset K_\beta$$

Fix $\varepsilon = \frac{1}{4} \min \{ \bar\varepsilon, \varrho, \delta \}$

and choose $\tau \in C^\infty(\mathbb{R})$

$$\tau(s) = \begin{cases} 1, & |s-\beta| \leq \varepsilon \\ 0, & |s-\beta| \geq 2\varepsilon \end{cases}$$

and $\eta : X \to \mathbb{R}$ Lipschitz with $0 \leq \eta \leq 1$
and

$$\eta(u) = \begin{cases} 1, & u \notin \mathcal{N}_{\beta,\delta} \\ 0, & u \in \mathcal{N}_{\beta,\delta_-} \end{cases}$$

Set

$$e(u) = \begin{cases} -\tau(\mathcal{F}(u)) \cdot \eta(u) \cdot \tilde{e}(u), & u \in \tilde{X} \\ 0 & \text{else} \end{cases}$$

Afterwards we proceed as in Lemma
only to show (iv) is a bit more
complicated:

We have for $u \in \tilde{\mathcal{F}}_{\beta+\varepsilon}$ with $\tilde{\mathcal{F}}(\phi(u,1)) \geq \beta-\varepsilon$

$$\tilde{\mathcal{F}}(\Phi(u,1)) = \tilde{\mathcal{F}}(u) + \int_0^1 \frac{d}{dt}\tilde{\mathcal{F}}(\phi(u,t))\,dt$$

$$\leq \beta+\varepsilon - \int_0^1 \eta(\Phi(u,t)) \cdot \tau(\tilde{\mathcal{F}}(\phi(u,t))) \cdot \langle \tilde{e}(\phi(u,t)), \tilde{\mathcal{F}}'(\phi(u,t)) \rangle$$

$$\leq \beta-\varepsilon - \mathcal{L}^1(\{t : \phi(u,t) \notin \mathcal{N}_{\beta,\delta}\}) \cdot \frac{\delta}{2}.$$

If either $u \notin \mathcal{N}$ or $\phi(u,1) \notin \mathcal{N}$ we get

$$\mathcal{L}^1(\{t : \phi(u,t) \notin \mathcal{N}_{\beta,\delta}\}) \geq \frac{\delta}{2}$$

since have

distance

$$\rightsquigarrow \tilde{\mathcal{F}}(\phi(u,1)) < \beta+\varepsilon - \frac{\delta}{2}\delta$$

$$< \beta-\varepsilon$$

↯

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Theorem 6.11: Assume $\mathcal{F} \in C^1(X)$ admits a relative minimizer $u_0 \in X$ and there exists $u_1 \in X$ with $\mathcal{F}(u_1) \leq \mathcal{F}(u_0)$.
Let
$$\beta = \inf_{\gamma \in \Gamma} \sup_{0 \leq s \leq 1} \mathcal{F}(\gamma(s))$$
where
$$\Gamma = \{ \gamma \in C^0([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1 \}$$
and suppose that $\mathcal{F}$ sat. $(P\text{-}S)_\beta$.
Then either

(i) $\beta > \mathcal{F}(u_0)$ and $K_\beta \neq \emptyset$

(ii) $\beta = \mathcal{F}(u_0)$ and there is a $u \in K_\beta$ which is not a rel. minimizer

(iii) $\beta = \mathcal{F}(u_0) = \mathcal{F}(u_1)$ and u_0, u_1 can be connected in any neighborhood of K_β. //

Proof:

(i) If $\beta > \mathcal{F}(u_0)$, then u_0 & u_1 are in different components of $\mathcal{F}_\beta$

$\leadsto$ apply Thm.

(ii) Suppose $\mathcal{F}(u_0) = \beta$ and that K_β consists entirely of relative minimizers.

$\leadsto$ $\forall u \in K_\beta$ $\exists$ open neighborhood $V(u)$ such that

$$\mathcal{F}(v) \geq \mathcal{F}(u) = \beta \quad \forall v \in V(u)$$

For any given neighborhood $\mathcal{N}$ of K_β we set

$$\tilde{\mathcal{N}} := \mathcal{N} \cap \bigcup_{u \in K_\beta} V(u) \subset \mathcal{N}$$

$\leadsto$ $\tilde{\mathcal{N}} \subset \mathcal{N}$ is an open neighborhood of K_β with

$$\mathcal{F}(v) \geq \beta \quad \forall v \in \tilde{\mathcal{N}}.$$

Now we apply Thm. 6.10 to $\tilde{\mathcal{N}}$ with $\bar{\varepsilon} = 1$ to get $\Phi \in C^0(X \times [0,1], X)$ and $\varepsilon \in (0,1)$ with all the desired properties. Let $\gamma \in \Gamma$ be such that

$$\sup_{0 \leq s \leq 1} \mathcal{F}(\gamma(s)) < \beta + \varepsilon$$

and

$$\gamma_1 := \Phi(\gamma, 1)$$

Then

$$\gamma_1 \subset \mathcal{F}_{\beta-\varepsilon} \cup \tilde{\mathcal{N}}$$

and $\gamma_1(0) = u_0$ since $u_0 \in X_\beta$.
As $\mathcal{F}_{\beta-\varepsilon}$ and $\tilde{\mathcal{N}}$ are disjoint,
we get

$$\gamma_1 \subset \tilde{\mathcal{N}}$$

and if furthermore u_* was in X_β
and hence $\gamma_1(u_*) = u_1$ the Thm. would
be proven.

Claim: u_1 is critical.

Proof: If not, we can find a point $\bar{u}_1$
nearby such that

$$\mathcal{F}(\bar{u}_1) < \mathcal{F}(u_1) \le \beta = \mathcal{F}(0)$$

but still

$$\beta = \inf_{\gamma \in \bar{\Gamma}} \sup_{0 \le s \le 1} \mathcal{F}(\gamma(s))$$

$$= \mathcal{F}(u_0).$$

where $\bar{\Gamma} = \{ \gamma \in C^0([0,1], X) : \gamma(0) = u_0, \gamma(1) = \bar{u}_1 \}$

Thm 6.10 $\rightarrow$ $\exists\ \Phi, \varepsilon > 0$ with all the prop.
with $\bar{\varepsilon} = \beta - \mathcal{F}(\bar{u}_1)$

Take $\gamma \in \bar{\Gamma}$ with $\sup \mathcal{F}(\gamma(s)) < \beta + \varepsilon$

$\rightsquigarrow \gamma_1 := \Phi(\gamma, 1) \subset \mathcal{F}_{\beta-\varepsilon} \cup \tilde{\mathcal{N}}$ but

$\gamma_1(0) \in \tilde{\mathcal{N}}$, $\gamma_1(1) \in \mathcal{F}_{\beta-\varepsilon}$ $\lightning$

Def: Let M be a topological space, $\Phi \in C^0(M \times [0,1], M)$, $\Phi(\cdot, 0) = id$. A set $\mathcal{A} \in \mathcal{P}(M)$ or a family of sets $\mathcal{A}$ is Φ-invariant if

$$\Phi(F, t) \in \mathcal{A} \quad \forall F \in \mathcal{A},\ t \in [0,1];$$

Examples:

(i) $\mathcal{A} = \{M\}$ is Φ-invariant, whenever $\Phi(\cdot, t) : M \to M$ is surjective.

(ii) $\mathcal{A} = \{ \{u\} : u \in M \}$

(iii) Let $\mathcal{F}_0 \in C^1(X)$ be such that $\mathcal{F}_0$ has two connected comp. with u_1, u_0 in different comp.
let
$$\alpha = \max\{\mathcal{F}(u_0), \mathcal{F}(u_1)\} < 0$$
and
$$\Gamma := \{ \gamma \in C^0([0,1] : X) \\ \gamma(0) = u_0,\ \gamma(1) = u_1 \}.$$

Then Γ is invariant with resp. to any $\Phi \in C^1(X \times [0,1], X)$ with $\Phi(\cdot, 0) = id$ and
$$\phi(u, t) = u \quad \forall u \in \mathcal{F}_{\alpha/2}$$

vi) Let $\mathcal{F} \in C^1(X)$, $\alpha < \beta$ and suppose that

$$\beta_k = \inf_{\gamma \in \Pi_k} \sup_x \mathcal{F}(\gamma(x)) \geq \beta$$

where

$$\Pi_k := \{ \gamma \in C^0(\overbrace{B_1(0, \mathbb{R}^k)}^{=: B_k}, X) : \gamma|_{\partial B_k} = \gamma_0 : \partial B_k \to \mathcal{F}_\alpha \}$$

Then Π is Φ invariant w.r.t Φ^k ∇

Theorem 6 : (Minimax-principle, Palais's)

Let $\mathcal{F} \in C^1(X)$ sat. (P-S)$_\beta$ for every $\beta \in \mathbb{R}$ and let $\mathfrak{A}$ be Φ invariant for all Φ satisfying properties (i) - (iii) of Thm.

Let

$$\beta_{\mathfrak{A}} := \inf_{A \in \mathfrak{A}} \sup_{A} \mathcal{F}$$

If $|\beta_{\mathfrak{A}}| < \infty$ we get $K_{\beta_{\mathfrak{A}}} \neq \phi$. //

Proof: Assume $|\beta_{\mathfrak{A}}| < \infty$ but $K_{\beta_{\mathfrak{A}}} = \phi$. Let $\varepsilon > 0$, $\phi \in C^0(X \times [0,1], X)$ be as in Thm. for $\bar{\varepsilon} = 1$, $N = \phi$. As $|\beta_{\mathfrak{A}}| < \infty$, there is an $A \in \mathfrak{A}$ with

$$\sup_{A} \mathcal{F} < \beta_{\mathfrak{A}} + \varepsilon$$

$\rightsquigarrow$ $A_1 := \Phi(A, 1) \in \mathfrak{A}$ but $\sup_{A_1} \mathcal{F} < \beta_{\mathfrak{A}} - \varepsilon$ ↯