§ 3.2 Constraints

Using constraints we can show

Theorem 3. : For $p \in (2, 2^*)$ there is a non-trivial weak solution u of

$$(3.5) \begin{cases} \Delta u = -|u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

i.e. $u \neq 0$. //

Proof: (i) We get this solution by minimizing

$$\tilde{\mathcal{F}} : M \to \mathbb{R}$$
$$\tilde{\mathcal{F}}(u) = \int \frac{1}{2} |\nabla u|^2 \, dx$$

on the set

$$M := \left\{ u \in \overset{\circ}{W}{}^{1,2}(\Omega) : g(u) = \int |u|^p dx = 1 \right\}.$$

We know from previous discussions, that $\tilde{\mathcal{F}}$ is coercive and (s. w. l.s.c). Furthermore, Rellich - Kondrakov tells us that for $u_m \to u$ weakly in $\overset{\circ}{W}{}^{1,2}(\Omega)$, $u_m \in M$ we have $u_m \to u$ in $L^p(\Omega)$ (note that $p < 2^*$!)

and hence $\int |u|^p dx = \lim_{n\to\infty} \int |u_n|^p dx = 1$. So $u \in M$ and thus we have shown that M is weakly closed.

Thm 3.2 $\rightsquigarrow$ $\tilde{\mathcal{F}}$ is bounded from below and attains its minimum on M, i.e. there is an $u \in M$ with

$$\tilde{\mathcal{F}}(u) = \inf_M \tilde{\mathcal{F}}.$$

(ii) As $\mathcal{F}, \mathcal{G}$ are C^1 on $W_0^{1,2}(\Omega)$ we get (as in the section about Lagrange multipliers) that there is a $\lambda \in \mathbb{R}$ such that

$$\mathcal{F}'(u) - \lambda\, \mathcal{G}'(u) = 0$$

i.e.

$$\int \left(\nabla u \nabla h - \lambda |u|^{p-2} u \cdot h \right) dx = 0 \quad \forall h \in W_0^{1,2}(\Omega)$$

i.e. u solves weakly

$$+\Delta u + \lambda |u|^{p-2} u = 0 \text{ in } \Omega.$$

(iii) Testing with u we get

$$\int |\nabla u|^2 dx - \lambda = 0$$

i.e. $\lambda = \int |\nabla u|^2 dx > 0.$

Setting $\tilde{u} := \lambda^{1/2-p} u$ we get

$$\begin{cases} -\Delta \tilde{u} = + |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$ //

The situation changes dramatically, if we assume that $p \geq 2^*$, i.e. for critical and supercritical exponents.

<u>Theorem 3.</u> : Suppose $\Omega \neq \mathbb{R}^n$ is a smooth (possibly unbounded) domain which is strictly star-shaped with respect to the origin in $\mathbb{R}^n$ and let $\lambda \leq 0$. Then any solution $u \in \overset{\circ}{W}{}^{1,2}(\Omega) \cap C^2(\Omega) \cap C^1(\bar{\Omega})$ of

$$\begin{cases} -\Delta u = |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

vanishes identically. //

The proof is based on the "Pohozaev identity":

Lemma 3. : Let $g : \mathbb{R} \to \mathbb{R}$ be continuous and let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a solution to

$$\begin{cases} -\Delta u = g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

in a domain $\Omega \subset\subset \mathbb{R}^n$. Then

$$\frac{n-2}{2} \int_\Omega |\nabla u|^2 \, dx - n \int_\Omega G(u) \, dx$$

$$+ \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 x \cdot \nu \, d\sigma = 0$$

where ν denotes the exterior unit normal. //

Proof of Thm 3. : We get from Lemma

$$\frac{n-2}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{n}{p} \int_\Omega |u|^p \, dx$$

$$+ \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 x \cdot \nu \, d\sigma = 0.$$

and testing with u we get

$$0 = \int_\Omega \left(-\Delta u \, u - |u|^p \right) dx$$

$$= \int_\Omega \left(|\nabla u|^2 - |u|^p \right) dx.$$

hence

$$\left(\frac{n-2}{2} - \frac{n}{p}\right) \int_\Omega |\nabla u|^2$$

$$+ \frac{1}{2} \int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 \underbrace{x\cdot\nu}_{>0}\, dx = 0.$$

If now $p > \frac{n-2}{2n}$ we get $\int_\Omega |\nabla u|^2 dx = 0$. If $p = \frac{n-2}{2n} = 2^*$, we can still deduce

$$\int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 dx = 0$$

which implies $u \equiv 0$ by the principle of unique continuation.

Proof of Lemma :

We have

$$0 = (\Delta u + g(u))\,(x\cdot\nabla u)$$

$$= \operatorname{div}(\nabla u\,(x\cdot\nabla u)) - |\nabla u|^2 - x\cdot\nabla\left(\frac{|\nabla u|^2}{2}\right) + x\cdot\nabla G(u)$$

$$= \operatorname{div}\left(\nabla u\,(x\cdot\nabla u) - x\,\frac{|\nabla u|^2}{2} + x\,G(u)\right)$$

$$+ \frac{n-2}{2}|\nabla u|^2 - n\,G(u)$$

Integrating over Ω and using

$$x\cdot\nabla u = x\cdot\nu\,\frac{\partial u}{\partial\nu} \quad \text{on } \partial\Omega$$

as $u=0$ on $\partial\Omega$, we get the lemma.

$\square$

Remarks:

(i) We will see later on that on topologically complicated domains and $p = 2^*$ the critical exponent, there are non-trivial smooth solutions to

$$\begin{cases} -\Delta u = |u|^{2^*-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

($\rightsquigarrow$ Bahri & Coron)

(ii) For $p > 2^*$ the analog to 2^* seems to be widely open (and much more complicated)

Plateau's Problem:

Let $\Gamma \subset \mathbb{R}^3$ be a smooth Jordan curve. Experiments convinced Plateau that every such curve is spannend by a surface of least area

Model: Surface ~~is~~ has the topological type of a disc

$$D = \{ z = (x, y) \in \mathbb{R} : x^2 + y^2 < 1 \}$$

$\boxed{\text{(naive) approach:}}$ Minimize area

$$A(u) = \int_\Omega \sqrt{\det (\nabla u^t \nabla u)}\, dz$$

$$= \int_\Omega \sqrt{|u_x|^2 |u_y|^2 - (u_x \cdot u_y)^2}\, dz$$

among "surfaces" $u \in H^{1,2} \cap C^0(\bar{\Omega}, \mathbb{R}^3)$ satisfying the Plateau boundary condition

$u|_{\partial\Omega} : \partial\Omega \to \Gamma$ is a monotone parametr. of Γ preserving the orientation.

Problem: $\mathcal{A}$ invariant under changes of the parametrizations

Idea: [Douglas & Rado 1930]

There is a deep connection to the Dirichlet energy: If u is conformal, i.e. $u_x \cdot u_y = 0$, $|u_x|^2 = |u_y|^2$ we have

$$\mathcal{A}(u) = \int_\Omega \sqrt{|u_x|^2 |u_y|^2 - (u_x \cdot u_y)^2}\, dz$$

$$= \frac{1}{2} \int_\Omega \left(|u_x|^2 + |u_y|^2 \right) dz = \mathcal{D}(u).$$

Actually we even have

Lemma: $\inf\limits_{u \in \mathcal{C}(\Omega)} \mathcal{A}(u) = \inf\limits_{u \in \mathcal{C}(\Omega)} \mathcal{D}(u)$

Plateau's problem is then reduced to

Theorem 3. : For any C^1- embedded curve Γ there exists a minimizer of the Dirichlet energy $\mathcal{D}$ in $\mathcal{C}(\Gamma)$ //

To prove this theorem we still have to deal with conformal invariance of $\mathcal{D}$, i.e. that

$$\mathcal{D}(u) = \mathcal{D}(u \circ g) \quad \forall g \in \mathcal{G}$$

where

$$\mathcal{G} = \Big\{ g : z \to g(z) = e^{i\phi} \frac{a+z}{1-\bar{a}z}, \; a \in \mathbb{C}, \; |a| \leq 1, \; 0 \leq \phi < 2\pi \Big\}$$

(the conformal group of Möbius transformations of Ω.

Solution: Use a three point condition ! We use fixing the image of three points determines a unique $g \in \mathcal{G}$!

So we fix a parametization

$$\gamma : \partial\Omega \to \Gamma \quad (\in C^1)$$

and let

$$\mathcal{C}^*(\Gamma) = \{ u \in \mathcal{C}(\Gamma) : u(e^{2\pi i k/3}) = \gamma(e^{2\pi i k/3}), \; k = 1, 2, 3 \}$$

Now we are set to prove Theorem

Proof of Theorem:

Let $u_m \in \mathcal{C}(\Gamma)$ be a minimizing sequence of $\mathcal{D}$, i.e.

$$\lim_{m\to\infty} \mathcal{D}(u_m) = \inf_{\mathcal{C}(\Gamma)} \mathcal{D} .$$

(i) First we will change this minimizing sequence to a nicer one.
There is a unique $g \in \mathcal{G}$ with

$$g_k(2^{2\pi i k/3}) = f_k^{-1} \circ \gamma(2^{2\pi i k/3})$$

Where $f_k = u_m|_{\partial\Omega}$.

Then $\tilde{u}_m := u_m \circ g_m$

is still a minimizing sequence and

$$\tilde{u}_m(e^{2\pi i k/3}) = \gamma(e^{2\pi i k/3}),$$

i.e. $\tilde{u}_m \in \mathcal{C}^*(\Gamma)$.

(ii) We can assume that

$$\mathcal{E}(u_m) = \inf_{\substack{v \in W^{1,2} \cap C^0 \\ v = u_m \text{ on } \partial\Omega}} \mathcal{E}(v).$$

If not, we exchange u_m by this minimizer, which belongs to $C^\infty(\Omega) \cap C^0(\bar{\Omega})$ and solves

$$\Delta u_m = 0 \quad \text{in } \Omega.$$

(iii) <u>Claim:</u> $f_m := \gamma^{-1} \circ u_m|_{\partial\Omega} : \partial\Omega \to \mathbb{R}^3$

is uniformly continuous. //