§ 5 Concentration compactness

Idea: Study how compactness fails $\overset{\nabla}{\circ}$

(Problems with ∞ $\overset{\nabla}{\circ}$)

Example 1: For $a \in C^0(\mathbb{R}^n)$ with

$$a(x) \to a_\infty > 0 \text{ as } |x| \to \infty$$

and $2 < p < 2^* = \frac{2n}{n-2}$, if $n \geq 3$

we consider the equation

$$(5.1) \quad -\Delta u + a(x) u = u|u|^{p-2} \text{ on } \mathbb{R}^n$$
$$u(x) \to 0 \quad \text{as} \quad |x| \to \infty.$$

We can try to find a solution minimizing

$$\tilde{F}(u) = \frac{1}{2} \int_{\mathbb{R}^n} \left(|\nabla u|^2 + a(x) u^2 \right) dx$$

on

$$M = \{ u \in W^{1,2}(\mathbb{R}^n) \mid \|u\|_{L^p} = 1 \}$$

Claim:

$\tilde{F}$ is coercive and (s.w.l.s.) on M but M is $\boxed{\text{not}}$ weakly sequentially closed

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(i) To see this, we let

$$\Omega := \left\{ x \in \mathbb{R}^n : \ a(x) \leq \frac{1}{2} a_\infty \right\}$$

$$\subset\subset \mathbb{R}^n.$$

Then

$$F(u) \geq \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx$$

$$+ \frac{1}{4} a_\infty \int_{\mathbb{R}^n} |u|^2 \, dx$$

$$\underbrace{+ \frac{1}{2} \int_\Omega \underbrace{\left(a(x) - \frac{a_\infty}{2} \right)}_{\| \ldots \|_{L^\infty} \leq C < \infty} u^2 \, dx}_{\leq C \, \|u\|^2_{L^p} \leq \hat{C} < \infty}$$

$$\geq \min\left\{ \frac{1}{2}, \frac{a_\infty}{4} \right\} \|u\|^2_{W^{1,2}}$$

$$- C.$$

$\rightsquigarrow$ F is coercive.

(ii) Furthermore,

$$\tilde{\mathcal{F}}(u) = \frac{1}{2}\int_{\mathbb{R}^n}\left(|\nabla u|^2 + \max\left\{a(x), \frac{a_\infty}{2}\right\}|u|^2\right) dx$$

$$+ \frac{1}{2}\int_{\mathbb{R}^n} \underbrace{\min\left\{0, a(x) - \frac{a_\infty}{2}\right\}}_{\operatorname{supp}(\cdot)\,\subset\,\Omega} u^2\,dx$$

$$=: \tilde{\mathcal{F}}^1(u) + \tilde{\mathcal{F}}^2(u).$$

Then $\tilde{\mathcal{F}}^1(u)$ is (s.w.l.s) as it is the square of a norm equiv. to $\|\cdot\|_{H^{1,2}}$

By Rellich's thm, $\tilde{\mathcal{F}}^2(u)$ is even weakly continuous.

(iii) For $u \in M$, $x_0 \in \mathbb{R}^n$ let

$$u_{x_0}(x) := u(x - x_0)$$

Then

$$u_{x_0} \xrightarrow{w} 0 \quad \text{as} \quad |x_0| \to \infty$$

(but $0 \notin M$). and

$$\tilde{\mathcal{F}}(u_{x_0}) = \frac{1}{2}\int_{\mathbb{R}^n}\left(|\nabla u|^2 + a(x - x_0)u^2\right)dx$$

$$\xrightarrow[\text{Lebesgue}]{(|x| \to \infty)} \frac{1}{2}\int_{\mathbb{R}^n}\left(|\nabla u|^2 + a_\infty u^2\right)dx$$

$$= \tilde{\mathcal{F}}_\infty(u)$$

the "functional at infinity".

$\square_{\text{Claim}}$

Let

$$I = \inf_{u \in M} \tilde{F}(u),$$
$$I_\infty = \inf_{u \in M} \tilde{F}_\infty(u).$$

Theorem 5.1 (P.-L. Lions)

(i) We have $I \leq I_\infty$. Moreover, if $I < I_\infty$, there exists a solution
$$0 \leq u \in W^{1,2}(\mathbb{R}^n), u \neq 0$$
of (5.1).

(ii) The condition $I < I_\infty$ is necessary and sufficient for the relative compactness of all minimizing sequence $(u_k) \subset M$ of F. //

Remarks:

(i) If $I = I_\infty$ there still might be a converging minimizing sequence.

(ii) Since for $u \in M$ we have

$$1 = \|u\|_{L^p}^p = \int_{\{|u| \geq 1\}} |u|^p \, dx + \int_{\{|u| < 1\}} |u|^p \, dx$$

$$\underset{2<p<2^*}{\leq} \int_{\{|u| \geq 1\}} |u|^{2^*} dx + \int_{\{|u|<1\}} |u|^2 dx$$

$$\underset{\text{Sobolev-inequality}}{\leq} C \left(\|u\|_{W^{1,2}}^{2^*} + \|u\|_{W^{1,2}}^2 \right)$$

and $(\mathcal{F}_\infty(\cdot))^{1/2}$ is equivalent to the norm $\|\cdot\|_{W^{1,2}}$, we get that $I_\infty > 0$.

(iii) For $0 \leq \lambda \leq 1$ we set

$$M_\lambda := \{ u \in W^{1,2}(\mathbb{R}^n) \mid \|u\|_{L^p}^p = \lambda \}$$

and

$$I_\lambda = \inf_{u \in M_\lambda} \mathcal{F}(u), \quad I_{\lambda,\infty} = \inf_{u \in M_\lambda} \mathcal{F}_\infty(u).$$

Claim: $I < I_\infty$ implies

$I < I_\lambda + I_{1-\lambda,\infty}$ for all $\lambda \in [0, 1)$. //

Proof: By the homogeneity

$$\tilde{\mathcal{F}}_\infty(u) = \tilde{\mathcal{F}}_\infty\left(\frac{u}{\|u\|_{L^p}}\right) \cdot \lambda^{2/p}$$

$$\forall\, u \in M_\lambda .$$

Hence,

$$I_\lambda = \lambda^{2/p} I \quad \& \quad I_{\lambda,\infty} = \lambda^{2/p} I_\infty$$

Case $I \leq 0$:

Then $I - I_\lambda = (1-\lambda^{2/p}) I \leq 0$

$$\underset{\substack{\uparrow \\ \text{Remark (ii)}}}{<} I_{1-\lambda,\infty}$$

Case $I > 0$: Then

$$I - I_\lambda = \underbrace{(1-\lambda^{2/p})}_{\leq (1-\lambda) \leq (1-\lambda)^{2/p}} I \leq (1-\lambda)^{2/p} I$$

$$\underset{\substack{1-\lambda>0 \\ I_\infty > I}}{<} (1-\lambda)^{2/p} I_\infty = I_{1-\lambda,\infty}$$

$\square$

Proof:

(i) For all $u \in M$, $x_k \in \mathbb{R}^n$, $|x_k| \to \infty$ we get

$$I \le \tilde{F}(u(\cdot - x_k)) \overset{k\to\infty}{\longrightarrow} \tilde{F}_\infty(u)$$

Hence,

$$I \le \inf_{u \in M} \tilde{F}_\infty(u) = I_\infty$$

(ii) Necessity: For $I = I_\infty$ we will construct a divergent minimizing sequence $(\tilde{u}_k) \subset M$ of $\tilde{F}$ as follows:

Let (u_k) be a minimizing sequence of $\tilde{F}_\infty$, i.e.

$$\lim_{k\to\infty} \tilde{F}_\infty(u_k) = I_\infty = I.$$

By (3) there exist an $(x_k \in \mathbb{R}^3$ such that

$$|\tilde{F}(u_{k, x_k}) - \tilde{F}_\infty(u_k)| < \frac{1}{k}$$

and

$$\tilde{u}_k = u_{k, x_k} \overset{\omega}{\longrightarrow} 0 \quad \text{as } (k \to \infty)$$

(iii) Sufficiency:

Let $(u_k) \subset H$ with $\mathcal{F}(u_k) \to I < I_\infty$ for $k \to \infty$.

$\mathcal{F}$ coercive $\leadsto$ (u_k) bounded and a subsequence converges weakly to u in $H^1(\mathbb{R}^n)$ and pointwise almost everywhere.

Lemma 5.2: $\mathcal{F}(u_k) = \mathcal{F}(u) + \mathcal{F}_\infty(u_k - u) + o(1)$

where $o(1) \to 0$ for $(k \to \infty)$ //

Proof: We see that

$$\|\nabla u_k\|_{L^2}^2 = \|\nabla((u_k - u) + u)\|_{L^2}^2$$

$$= \|\nabla(u_k - u)\|_{L^2}^2 + \|u\|_{L^2}^2 + \underbrace{2\int \underbrace{\nabla(u_k - u)\nabla u\,dx}_{\rightharpoonup 0}}_{= o(1)}$$

and

$$\int a(x)\,u_k^2\,dx = \int a(x)(u_k - u)^2\,dx$$

$$+ \int a(x)\,u^2 + \underbrace{2\int a(x)\,\underbrace{(u_k - u)}_{\rightharpoonup 0}\cdot u\,dx}_{= o(1)}$$

Claim: $\displaystyle\int_{\mathbb{R}^n} a(x)\,(u_k-u)^2\,dx$

$$= \int_{\mathbb{R}^n} a_\infty (u_k-u)^2\,dx + o(1).$$

Proof of the claim

For $\tilde{\varepsilon} > 0$ we decompose using $\Omega_{\tilde{\varepsilon}} = \{ |a_k(x) - a_\infty| > \tilde{\varepsilon} \}$

$$\left| \int_{\mathbb{R}^n} (a_k(x) - a_\infty)\,(u_k-u)^2\,dx \right|$$

$$\leq \int_{\mathbb{R}^n - \Omega_{\tilde{\varepsilon}}} |(a_k(x)-a_\infty)(u_k-u)^2|\,dx$$

$$+ \int_{\Omega_{\tilde{\varepsilon}}} |(a_k(x)-a_\infty)(u_k-u)^2|\,dx$$

$$\leq \tilde{\varepsilon}\, \overbrace{\|u_k - u\|^2_{L^2}}^{\leq C \text{ (independent of } k\tilde{\varepsilon})} + o(1)$$

as Rellich's theorem tells us that

$$u_k \to u \quad \text{in } L^2(\Omega_{\tilde{\varepsilon}}).$$

For $\varepsilon > 0$ we let $\tilde{\varepsilon} = \frac{\varepsilon}{2C}$ and get

$$\left| \int_{\mathbb{R}^n} (a_k(x) - a_\infty)(u_k-u)^2\,dx \right|$$

$$\leq \varepsilon/2 + o(1) \leq \varepsilon \text{ for } k \text{ large}$$

□

Lemma 5.3: $1 = \|u_k\|^p_{L^p} = \|u_k - u\|^p_{L^p} + \|u\|^p_{L^p} + o(1)$. //

Proof: $f_k(t) := |u_k - tu|^{p-2}(u_k - tu)u$

(i) Using the fundamental theorem of calculus we get

$$|u_k|^p - |u_k - u|^p = -\int_0^1 \frac{d}{dt}(|u_k - tu|^p)\,dt$$

$$= -p\int_0^1 |u_k - tu|^{p-2}(u_k - tu)\,u\,dt$$

$$\to p\int_0^1 (1-t)^{p-1}\,dt\,|u|^p = |u|^p$$

Lebesgue a.e.

(ii) Furthermore,

$$\int_\Omega \left(\int |f_k(t)|\,dt\right)dx \le \int_\Omega (|u_k| + |u|)^{p-1}\,u\,dx$$

$$\underset{\text{Hölder}}{\le} \underbrace{\left(\|u_k\|^{p-1}_{L^p(\Omega)} + \|u\|^{p-1}_{L^p(\Omega)}\right)}_{\le C} \underbrace{\|u\|_{L^p(\Omega)}}_{\to 0 \text{ as } \mathcal{L}^n(\Omega) \to 0}$$

$\to 0$ as $\mathcal{L}^n(\Omega) \to 0$ (uniformly in k)

Vitali $\leadsto$ $\|u_k\|^p_{L^p} - \|u_k - u\|^p_{L^p} \xrightarrow{k\to\infty} \|u\|^p_{L^p}$ □

Conclusion of the proof of Thm 5.1.

We have

$$\bar{I} = \mathcal{F}(u_k) + o(1)$$

$$\underset{\text{Lemma 5.2}}{=} \mathcal{F}(u) + \mathcal{F}_\infty(u_k - u) + o(1)$$

$$\underset{\text{Lemma 5.3}}{\geq} \bar{I}_\lambda + \bar{I}_{1-\lambda,\infty} + o(1).$$

where $\lambda = \|u\|_{L^p}^p \in [0,1]$.
So

$$\bar{I} \geq \bar{I}_\lambda + \bar{I}_{1-\lambda,\infty}$$

Thus $\lambda = 1$ by Remark, part ii ∇
and hence $u \in M$.

It follows that

$$\mathcal{F}(u) \leq \lim_{k \to \infty} \mathcal{F}(u_k) \leq \mathcal{F}(u)$$

hence, u is a minimizer. ∇