Functional Analysis
5th Problem Sheet - WS 2005/06

Please hand in your solutions until Wednesday, December 7, 2005, 1:00 pm.

Exercise 17
Let $X$ be a normed space and $S, T : X \to X$ linear operators satisfying $ST - TS = I$. Then: $S$ is not continuous or $T$ is not continuous.

Hint: First show that $ST^{n+1} - T^{n+1}S = (n + 1)T^n$ ($n \in \mathbb{N}_0$).

Exercise 18 (C)

a) Let $X$ be a Banach space, $X \neq \{0\}$, and let $K \in L(X)$ be defined by $K = 2I$. Show that the corresponding Neumann series is divergent, nevertheless the inverse $(I - K)^{-1}$ exists and is continuous on $X$.

b) Let $X$ be a normed space, $K \in L(X)$ and $y \in X$. Furthermore let the Neumann series $\sum_{n=0}^{\infty} K^n y$ converge to $x \in X$. Prove: $(I - K)x = y$.

c) Let $X$ be a normed space and $K \in L(X)$. Show that the following assertions are equivalent:

(a) $(I - K)$ is injective and $\sum_{n=0}^{\infty} K^n y$ converges for all $y \in (I - K)(X)$.

(b) $K^n \to 0$ pointwise on $X$.

Hint: $(I - K)x = y \implies x = y + Ky + \ldots + K^{n-1}y + K^n x$. 

- please turn over -
Exercise 19

a) Let $C^1([0,1])$ be endowed with the norms

$$
\|f\|_1 := \max \{\|f\|_{\infty}, \|f'\|_{\infty}\}
$$

and

$$
\|f\|_2 := \|f\|_{\infty} + \|f'\|_{\infty}
$$

for $f \in C^1([0,1])$, respectively.

Show that both norms are equivalent, and that $(C^1([0,1]), \|\cdot\|_2)$ is a Banach algebra. Is that true for $(C^1([0,1]), \|\cdot\|_1)$?

b) Let $(\mathcal{A}, \|\cdot\|)$ be a normed algebra with unit $e \neq 0$. Show that there exists a norm $|\cdot|$ equivalent to $\|\cdot\|$ such that $|e| = 1$.

*Hint:* For $x \in \mathcal{A}$ define operators $L_x \in L(\mathcal{A})$ with $L_x r := xr$ for $r \in \mathcal{A}$ and consider the mapping $x \mapsto Tx := L_x$. Then set $|x| := \|L_x\|.$

Exercise 20 (C)

Let $k : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function. Show that the mapping

$$
V : \left\{
\begin{array}{ccl}
C([0,1]) & \to & C([0,1]) \\
f & \mapsto & Vf
\end{array}
\right.
$$

given by

$$
(Vf)(s) = \int_0^s k(s,t)f(t)dt \quad \text{for all } s \in [0,1]
$$

defines a compact operator on $(C([0,1]), \|\cdot\|_{\infty})$. Moreover determine $\lim_{n \to \infty} \|V^n\|^{1/n}$.

$V$ is called Volterra integral operator.

Show that the Fredholm integral operator given in exercise 12 is a compact operator on $(C([0,1]), \|\cdot\|_{\infty})$ as well.

*Hint:* Use the theorem of Arzelà-Ascoli.