Please hand in your solutions until Wednesday, December 21, 2005, 1:00 pm.

Exercise 25
Let $X$ denote a $K$-vector space and let $p : X \to [0, \infty)$ be a seminorm on $X$.
Prove:

a) $Y := \{x \in X : p(x) = 0\}$ is a subspace of $X$.

b) $\hat{p}(x + Y) := p(x)$ defines a norm on $\hat{X} := X/Y$.

Exercise 26 (C)

a) Let $(X, \|\cdot\|)$ be a normed space (real or complex). Moreover let $x_1, \ldots, x_n \in X$ be linearly independent and $\alpha_1, \ldots, \alpha_n \in K$. Show that there exists a linear and continuous function $f : X \to K$, such that $f(x_k) = \alpha_k$ for $(k = 1, \ldots, n)$.

b) Let $X, Y$ be normed spaces, $X \neq \{0\}$. Show that, if $L(X, Y)$ is a Banach space, then $Y$ is complete. (This is the reverse statement of Theorem 4.2 (2).)

Exercise 27 (C)

Let $l_\infty$ denote the real vector space of all bounded real sequences. Show that there exists a continuous linear functional $L$, such that:

a) $\liminf_{k \to \infty} \xi_k \leq L(x) \leq \limsup_{k \to \infty} \xi_k$ for all $x = (\xi_k) \in l_\infty$,

b) $L(\xi_1, \xi_2, \xi_3, \ldots) = L(\xi_2, \xi_3, \xi_4, \ldots)$ for all $x = (\xi_k) \in l_\infty$.

Remark: Such a functional is called Banach limit.

Hint: Consider the function $p(x) = \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \xi_k$. Check that $p$ is sublinear and use the extension principle of Hahn-Banach (with a suitable subspace of $l_\infty$).
Exercise 28
Let $U, V$ and $W$ be closed subspaces of the normed space $X$. Show that:

a) $W = W^\perp$.
b) $U \cap V = (U^\perp + V^\perp)$.
c) $U^\perp \cap V^\perp = (U + V)^\perp$.
d) $U + V$ is closed if and only if $U + V = (U^\perp \cap V^\perp)$.

*Hint for a):* Theorem 11.3.

Remarks on the course in Functional Analysis:

- The exam ("studienbegleitende Prüfung") will take place on March 1, 2006, 10-12 am, location: Gaede-Hörsaal (30.22).
- You will get more information concerning the exam in time.