Please hand in your solutions until Wednesday, January 18, 2006, 1:00 pm.

Exercise 33 (C)

a) Let \( f : [0, 1] \to \mathbb{R} \) be defined by
\[
f(t) := \begin{cases} 
1, & t \in [0, 1] \cap \mathbb{Q} \\
0, & t \in [0, 1] \setminus \mathbb{Q}.
\end{cases}
\]
Show that there does not exist a sequence of continuous functions \( g_n : [0, 1] \to \mathbb{R} \), such that \( (g_n) \) converges pointwise to \( f \).

b) Let \( f : [0, \infty) \to \mathbb{R} \) be a continuous function satisfying
\[
\lim_{n \to \infty} f(nt) = 0 \quad (t \geq 0).
\]
Show that \( \lim_{t \to \infty} f(t) = 0 \).

Hint: Baire’s category principle. In a) consider the sets
\[
F_n := \left\{ t \in [0, 1] : |g_n(t) - g_m(t)| \leq \frac{1}{3} \text{ for all } m \geq n \right\}.
\]

Exercise 34

a) Show that completeness and countability are essential in the formulation of the Baire category principle.

b) Let \( a, b \in \mathbb{R} \), \( a < b \) and \( J := [a, b] \setminus \mathbb{Q} \). Show that \( J \) can not be represented as union of countable many closed subsets \( F_n \).

c) Show by means of part b), that there is no function \( f : [0, 1] \to \mathbb{R} \), such that \( f \) is continuous in each \( x \in [0, 1] \cap \mathbb{Q} \) and not continuous in each \( x \in [0, 1] \setminus \mathbb{Q} \).
Exercise 35
Let \((X, \|\cdot\|_1)\) be a Banach space. Prove:

a) Each norm \(\|\cdot\|_2\) on \(X\) with the properties
   
   (i) \(\|\cdot\|_1\) is stronger than \(\|\cdot\|_2\),

   (ii) \((X, \|\cdot\|_2)\) is a Banach space

   is equivalent to the norm \(\|\cdot\|_1\).

b) Each norm \(\|\cdot\|_2\) on \(X\), with the properties

   (i) \((X, \|\cdot\|_2)\) is a Banach space

   (ii) \(\forall x, y \in X, x \neq y, \exists f \in (X, \|\cdot\|_1)' \cap (X, \|\cdot\|_2)': f(x) \neq f(y)\)

   is equivalent to the norm \(\|\cdot\|_1\).

c) Let the linear functional \(\varphi\) be not continuous with respect to \(\|\cdot\|_1\). Then there is no norm \(\|\cdot\|_2\) on \(X\), such that

   (i) \((X, \|\cdot\|_2)\) is a Banach space,

   (ii) \(\varphi\) is continuous with respect to \(\|\cdot\|_2\), and

   (iii) all \(f \in (X, \|\cdot\|_1)'\) are continuous with respect to \(\|\cdot\|_2\).

Exercise 36 (C)
Let \((X, \|\cdot\|_1), (Y, \|\cdot\|_2)\) be normed spaces, \(D\) a subspace of \(X\) and let \(A : D \rightarrow Y\) be a closed operator. Show that:

a) If \(A\) is injective, then \(A^{-1}\) is closed as well.

b) \(\|x\|_A := \|x\|_1 + \|Ax\|_2\) for all \(x \in D\) defines a norm on \(D\).

   We set \(D_A := (D, \|\cdot\|_A)\).

c) If \(X\) and \(Y\) are complete, then \(D_A\) is complete and \(A : D_A \rightarrow Y\) is continuous.

d) If \(X, Y\) are complete and \(A : D \rightarrow Y\) is bijective, then \(A^{-1} : Y \rightarrow D\) is continuous.

Remarks on the exam in Functional Analysis

- The exam („studienbegleitende Prüfung“) will take place on March 1, 2006, 10-12 am, location: Gaede-Hörsaal (30.22).

- You can announce for the exam from January 16, 2006, until February 24, 2006, in the secretary’s office of Prof. Plum (room 305).

  Later announcements will not be accepted!

- To announce bring along your exam approval („Zulassung zur Prüfung“).