Please hand in your solutions until Wednesday, February 1, 2006, 1:00 pm.

**Exercise 41 (C)**
Let $X, Y$ be Banach spaces. Moreover let $A : X \to Y$ be linear, $A \neq 0$, $N(A)$ closed in $X$, $A(X)$ closed in $Y$ and let the minimal modulus $\gamma(A)$ be positive. Show that:

- a) $A$ is continuous.
- b) If in addition $A$ is injective, then $\gamma(A) = \|A^{-1}\|^{-1}$.

**Hints:** Canonical injection, inverse mapping theorem.

**Exercise 42 (C)**

a) Let $V$ be a vector space over a field $K$. Moreover let $\varphi_1, \ldots, \varphi_n : V \to K$ be linear functionals, which are linearly independent. Show that the subspace $U := \bigcap_{k=1}^n N(\varphi_k)$ has codimension $n$.

b) Let $X$ be a normed space, $\dim X = \infty$. Show that there exists a subspace $U$ of $X$, such that $\text{codim } U = 1$ and $U$ is not closed.

**Definition:** Let $X$ be a normed space. A set $\Phi$ of linear functionals on $X$ is said to be **total** if from $\varphi(x) = 0$ for all $\varphi \in \Phi$ it follows that $x = 0$.

**Exercise 43**

a) Let $N$ be a countably infinite set. Show that there exists an uncountable index set $J$ and there exist subsets $U_\alpha \subseteq N$ ($\alpha \in J$), such that

- (i) $\text{card}(U_\alpha) = \infty$ ($\alpha \in J$) and
- (ii) $\text{card}(U_\alpha \cap U_\beta) < \infty$ ($\alpha, \beta \in J, \alpha \neq \beta$).

b) Prove: No total subset of $(l^\infty/c_0)'$ is countable.

**Hints:** In b) choose $N = \mathbb{N}$, $J$ and $(U_\alpha)_{\alpha \in J}$ as in a). Then set $f_\alpha = \chi_{U_\alpha}$, where $\chi_{U_\alpha}$ (considered as element of $l^\infty$) denotes the characteristic function of $U_\alpha$. Use also exercise 24 a).

- please turn over -
Exercise 44

a) Show that $c_0$ is proximinal in $l^\infty$, that means that to each $(x_n) \in l^\infty/c_0$ there exists $(z_n) \in c_0$ such that

$$ \|(x_n)\| = \inf_{(y_n) \in c_0} \|(x_n) - (y_n)\| = \|(x_n) - (z_n)\| $$

b) Show that $c_0$ is not a complementable subspace of $l^\infty$.

*Hint:* For a) use exercise 24 a), for b) exercise 43 and theorem 14.8 of the lecture course.

**Theorem 14.8:** Let $X, Y$ be Banach spaces and $A \in L(X, Y)$. Then $A$ is bijective if and only if $A'$ is bijective.