Please hand in your solutions until Wednesday, February 8, 2006, 1:00 pm.

Exercise 45
Let $X, Y$ be normed spaces, $X \neq \{0\}$. Moreover let $\Phi : L(X, Y) \to L(Y', X')$ be defined by $\Phi(A) = A'$. Show that: $\Phi$ is surjective if and only if $Y$ is reflexive, and in this case $\Phi$ is an isometric isomorphism.

Exercise 46 (C)
Let $1 < p < \infty$. Prove the following assertions:

a) A sequence $(x^{(n)})$ in $l^p$ converges weakly to $x \in l^p$ if and only if $(x^{(n)})$ is bounded (with respect to $\| \cdot \|_p$) and converges to $x$ coordinatewise.

b) There exists a weakly convergent sequence in $l^p$, which is not norm-convergent.

c) There exists a coordinatewise convergent sequence in $l^p$, which is not weakly convergent.

Hint: Banach-Steinhaus in a).

Exercise 47

a) Let $X$ be a Banach space and $M$ be a subset of $X'$. Show that $M$ is bounded if and only if $\{ \varphi(x) : \varphi \in M \}$ is bounded for all $x \in X$.

b) Let $X, Y$ be Banach spaces and let $(x_n)$ be a sequence in $X$, which is weakly convergent to $x \in X$. Show that:

(i) If $T \in L(X, Y)$, then $(Tx_n)$ converges weakly to $Tx$ in $Y$.

(ii) If $K \in L(X, Y)$ is compact, then $\lim_{n \to \infty} \|Kx_n - Kx\| = 0$. 

- please turn over -
Exercise 48 (C)
Prove the following assertions:

a) The spaces $l^p$ are reflexive for $1 < p < \infty$.

b) $c_0$, $l^1$ and $l^\infty$ are not reflexive.

c) The spaces $c_0$ and $l^p$ for $1 \leq p < \infty$ are separable.

d) $l^\infty$ is not separable.

Hints: For b) show first that $c_0$ is not reflexive. For d) show that there exists an uncountable subset $M$ of $l^\infty$ such that

$$\forall x, y \in M, \ x \neq y \ : \ |x - y|_\infty = 1.$$  \hfill (1)

Then by means of (1) try to prove via contradiction.