Exercise 49 (C)
Let \((Z, \| \cdot \|)\) be a normed space satisfying the parallelogram law
\[
\|x + y\|^2 + \|x - y\|^2 = 2 \|x\|^2 + 2 \|y\|^2 \quad (x, y \in Z).
\]
Show that \(Z\) is an inner product space. In more detail, prove:

a) If \(Z\) is a real normed space, then
\[
(x, y) := \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2) \quad (x, y \in Z)
\]
is an inner product on \(Z\), and \(\|x\| = (x, x)^{1/2}\) for all \(x \in Z\).

b) If \(Z\) is a complex normed space, then
\[
(x, y) := \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2 + i \|x + iy\|^2 - i \|x - iy\|^2) \quad (x, y \in Z)
\]
is an inner product on \(Z\), and \(\|x\| = (x, x)^{1/2}\) for all \(x \in Z\).

*Hint:* For a) use exercise 13.

Exercise 50
Prove the following assertions:

a) There exists no inner product on \(C([0, 1])\) generating the maximum norm.

b) On \(l^p (1 \leq p \leq \infty)\) one can only define an inner product generating the \(l^p\)-norm, if \(p = 2\).

c) The setting
\[
(x, y) := \sum_{k=1}^{\infty} x_k \overline{y}_k \quad (x = (x_k), y = (y_k))
\]
defines an inner product on \(l^p\) for \(1 \leq p \leq 2\), but not for \(p > 2\).

d) An inner product can be defined on each real or complex vector space.
Exercise 51

a) Let $H$ be a Hilbert space and $U, V$ be closed subspaces of $H$, such that $U \perp V$. Show that $U + V$ is direct and closed.

b) Let $U = \{(\xi_k) \in l^2 : \xi_{2k} = 0 (k \in \mathbb{N})\}$ and $V = \{(\eta_k) \in l^2 : \eta_{2k-1} = k\eta_{2k} (k \in \mathbb{N})\}$. Prove: $U$ and $V$ are closed subspaces of $l^2$, the sum $U + V$ is direct, but not closed. Moreover determine $U^\perp$.

Exercise 52 (C)

Let $C([-1,1])$ be endowed with the inner product

$$(f, g) := \int_{-1}^{1} f(x)\overline{g(x)}dx.$$ 

Moreover let

$U_1 = \{f \in C([-1,1]) : f(x) = 0 (x \leq 0)\}$

and

$U_2 = \{f \in C([-1,1]) : f(0) = 0\}$.

a) Determine $U_1^\perp$ and $U_2^\perp$.

b) Show that $U_1^{1\perp} = U_1$ and $C([-1,1]) \neq U_1 \oplus U_1^\perp$. 