The solution of the Gevrey smoothing conjecture for the fully nonlinear homogeneous Boltzmann equation

Tobias Ried

joint work with **Jean-Marie Barbaroux, Dirk Hundertmark, Semjon Vugalter**

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Outline

- **Homogeneous Boltzmann equation**
  Boltzmann collision operator, singular angular collision kernel, Maxwell's weak formulation, weak solutions
- **Gevrey spaces**
  fractional heat equation, Gevrey spaces
- **Gevrey smoothing for the homogeneous Boltzmann equation** (Maxwellian molecules)
  main results, strategy of the proof
- **Commutator estimates**
  estimates in Fourier space, a Gronwall argument, extracting L^∞ bounds from L^2 bounds
- **Conclusion**
  closing the induction

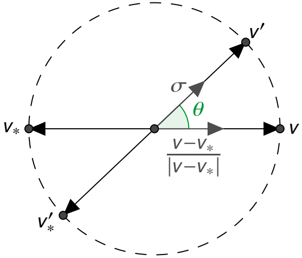
The Homogeneous Boltzmann Equation

- Time evolution of the distribution function $f : \mathbb{R}_+ \times \mathbb{R}^d \to [0, \infty)$ of a dilute gas in *spatially homogeneous* setting governed by

$$\partial_t f = Q(f,f) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} B\left(|v - v_*|, \tfrac{v-v_*}{|v-v_*|} \cdot \sigma\right) \left(f(v_*')f(v') - f(v_*)f(v)\right) \mathrm{d}\sigma \mathrm{d}v_*$$

- Binary, elastic collisions $\Longrightarrow$ conservation of momentum and energy

$$v' + v_*' = v + v_*$$
$$|v'|^2 + |v_*'|^2 = |v|^2 + |v_*|^2$$



- Parametrisation of pre-collisional velocities (σ-representation)

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v_*' = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma, \quad \cos\theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma, \quad \text{with} \quad \sigma \in \mathbb{S}^{d-1}$$

Important examples for the collision kernel B

- Inverse power-law interaction potential $\Phi(r) = r^{-(s-1)}$, $s > 2$:

$$B(|v - v_*|, \cos\theta) = |v - v_*|^\gamma\, b(\cos\theta), \quad \gamma = \frac{s - (2d-1)}{s-1}$$

where the angular collision kernel b has non-integrable singularity

$$\sin^{d-2}\theta\, b(\cos\theta) \sim \frac{\kappa}{\theta^{1+2\nu}} \quad \text{for } \theta \to 0, \quad \text{where } \nu = \frac{1}{s-1} \text{ for } d = 3.$$

- Maxwellian Molecules $B(|v - v_*|, \cos\theta) = b(\cos\theta)$
for instance $s = 5$ in the inverse power-law case in 3 dimensions
- Convenient and important: By replacing b with a symmetrized version, if necessary, we can w.l.o.g. assume $0 \le \theta \le \frac{\pi}{2}$

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Weak solutions and Maxwell's weak formulation

Natural assumptions on (weak) solutions: finite mass, energy, and entropy

$$f \in \mathscr{C}(\mathbb{R}_+; \mathscr{D}'(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}_+; L^1_2(\mathbb{R}^d) \cap L \log L(\mathbb{R}^d)), \quad f \geq 0, \quad f(0,\cdot) = f_0 \in L^1_2 \cap L \log L$$

Conservation Laws

- mass is conserved: $\int_{\mathbb{R}^d} f \, \mathrm{d}v = \int_{\mathbb{R}^d} f_0 \, \mathrm{d}v$
- kinetic energy is conserved: $\int_{\mathbb{R}^d} f \, v^2 \, \mathrm{d}v = \int_{\mathbb{R}^d} f_0 \, v^2 \, \mathrm{d}v$
- entropy is increasing: $-H(f) = -\int_{\mathbb{R}^d} f \log f \, \mathrm{d}v \geq -\int_{\mathbb{R}^d} f_0 \log f_0 \, \mathrm{d}v$

Weak formulation

For all $\varphi \in \mathscr{C}^1(\mathbb{R}_+; \mathscr{C}^\infty_0(\mathbb{R}^d))$ and for all $t \geq 0$ one has

$$\langle f(t,\cdot), \varphi(t,\cdot)\rangle - \langle f_0, \varphi(0,\cdot)\rangle - \int_0^t \langle f(\tau,\cdot), \partial_\tau \varphi(\tau,\cdot)\rangle \, \mathrm{d}\tau = \int_0^t \langle Q(f,f)(\tau,\cdot), \varphi(\tau,\cdot)\rangle \, \mathrm{d}\tau$$

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$$\langle Q(f,f)(\tau,\cdot),\varphi(\tau,\cdot)\rangle = \frac{1}{2}\int_{\mathbb{R}^{2d}}\int_{\mathbb{S}^{d-1}} b(\cos\theta)\, f(v_*)f(v)\,\left[\varphi(v')+\varphi(v'_*)-\varphi(v)-\varphi(v_*)\right]\,\mathrm{d}\sigma\mathrm{d}v\mathrm{d}v_*$$

- If $\varphi \in W^{2,\infty}(\mathbb{R}^d)$ then $\left|\int_{\mathbb{S}^{d-2}}\left[\varphi(v')+\varphi(v'_*)-\varphi(v)-\varphi(v_*)\right]\,\mathrm{d}\omega\right| \leq C_\varphi |v-v_*|^2\,\theta^2$
- $Q(f,f)$ well-defined if

$$\int_0^{\frac{\pi}{2}} b(\cos\theta)\,(1-\cos\theta)\,\sin^{d-2}\theta\,\mathrm{d}\theta = m_b < \infty$$

- m_b related to momentum transfer in scattering process

Existence (and Uniqueness) of weak solutions

Arkeryd 1981, Goudon 1997, Villani 1998, Mischler-Wennberg 1999, Toscani-Villani 1999, Desvillettes-Mouhot 2009,...

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Absence of Smoothing in the Grad Cut-off Case

- Simplification: Grad's angular cut-off assumption $\int_{\mathbb{S}^{d-1}} b(\cos\theta)\, \mathrm{d}\sigma = a < \infty$
- Splitting of the collision operator

$$Q(f,f) = \underbrace{Q^+(f,f)}_{\text{gain}} - \underbrace{Q^-(f,f)}_{\text{loss}} = Q^+(f,f) - f(Lf) \quad \text{where} \quad Lf = a\int_{\mathbb{R}^d} f(v)\,\mathrm{d}v.$$

- Homogeneous Boltzmann equation $\partial_t f + f(Lf) = Q^+(f,f)$

Duhamel Formula

$$f(t,v) = \mathrm{e}^{-\int_0^t Lf(\tau,v)\,\mathrm{d}\tau} f_0(v) + \int_0^t \mathrm{e}^{-\int_s^t Lf(\tau,v)\,\mathrm{d}\tau} Q^+(f,f)(s,v)\,\mathrm{d}s$$

$\Longrightarrow$ **Propagation of singularities (and regularity)!** [Mouhot-Villani 2004]

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Long-range interactions: singular angular collision kernels

- The situation is totally different if the angular collision kernel has a non-integrable singularity for small collision angles (*grazing collisions*)
- We consider the following type of singularity

$$\sin^{d-2}\theta\, b(\cos\theta) \sim \frac{\kappa}{\theta^{1+2\nu}} \quad \text{for } \theta \to 0,$$

for some $\kappa > 0$ and $0 < \nu < 1$. Not integrable near 0!

Observation

$Q(g,f)$ behaves like a singular integral operator with a leading term similar to a fractional Laplacian $(-\Delta)^\nu$.

Quantitatively: coercivity estimate [Alexandre-Desvillettes-Villani-Wennberg 2000]

$$-\langle Q(g,f), f\rangle \geq c_g \langle f, (-\Delta)^\nu f\rangle - l.o.t$$

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Intermezzo: Fractional Heat Equation

- Fractional heat equation ($\nu > 0$)

$$\begin{cases} \partial_t u + (-\Delta)^\nu u &= 0 \\ u|_{t=0} &= u_0 \in L^1(\mathbb{R}^d) \end{cases}$$

- in Fourier space

$$\widehat{u}(t,\xi) = e^{-t|\xi|^{2\nu}} \widehat{u_0}(\xi) \quad \text{with} \quad \widehat{u_0} \in L^\infty(\mathbb{R}^d),$$

so there exists a finite constant $M > 0$ such that

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Note that $G^\alpha(\mathbb{R}^d) \subset H^\infty(\mathbb{R}^d) = \cap_{s \geq 0} H^s(\mathbb{R}^d) \subset \mathscr{C}^\infty(\mathbb{R}^d)$.

- $0 < \alpha < 1$ ultra-analytic functions
- $\alpha = 1$ real analytic functions $\mathscr{C}^\omega$
- $\alpha > 1$ Gevrey-α functions G^α

} Gevrey spaces *interpolate between* $\mathscr{C}^\infty$ *and* $\mathscr{C}^\omega$

Heat equation $\partial_t u + (-\Delta)^\nu u = 0$ with initial condition in $L^1(\mathbb{R}^d)$

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Gevrey Smoothing Conjecture

Conjecture

Any weak solution of the non-cutoff homogeneous Boltzmann equation with a singular cross section kernel of order ν and with initial datum in $L^1_2(\mathbb{R}^d) \cap L \log L(\mathbb{R}^d)$, i.e., finite mass, energy and entropy, belongs to the Gevrey class $G^{\frac{1}{2\nu}}(\mathbb{R}^d)$ for strictly positive times.

That is, the homogeneous non-cutoff Boltzmann equation for Maxwellian molecules enjoys the same smoothing properties as the fractional heat equation.

[Desvillettes-Wennberg 2004]

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Known results

- **Existence** of Gevrey regular solutions for nice, in particular Gevrey, initial conditions [Ukai 1984]
- **Propagation** of Gevrey regularity [Desvillettes-Furioli-Terraneo 2009]

Smoothing Properties

- H^∞ (in particular $\mathscr{C}^\infty$) smoothing [Desvillettes-Wennberg 2004, Alexandre-ElSafadi 2004, Morimoto-Ukai-Xu-Yang 2009, Alexandre-Morimoto-Ukai-Xu-Yang 2012]
- Gevrey smoothing for the linearized Boltzmann equation [Morimoto-Ukai-Xu-Yang 2009, Lerner-Morimoto-Pravda-Starov-Xu 2014 (radially symmetric)]
- Similar results for the Kac equation, under some higher moments assumption [Lekrine-Xu 2009, Glangetas-Najeme 2013]
- Gevrey smoothing under very strong decay assumptions (Schwartz or even Maxwellian decay) [Morimoto-Ukai 2010,Yin-Zhang 2012, Lin 2014]

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Main Results

Theorem 1 [Barbaroux-Hundertmark-Ried-Vugalter 2015]

Let $d \geq 2$. Let f be a weak solution of the Cauchy problem

$$\partial_t f = Q(f, f), \qquad f|_{t=0} = f_0 \tag{hom. BE}$$

with initial datum $0 \leq f_0 \in L \log L(\mathbb{R}^d) \cap L^1_2(\mathbb{R}^d)$. Then, for all $0 < \alpha \leq \min\left\{\frac{\log(5/3)}{\log 2}, \nu\right\}$,

$$f(t, \cdot) \in G^{\frac{1}{2\alpha}}(\mathbb{R}^d)$$

for all $t > 0$.

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Main Results

Theorem 1 [Barbaroux-Hundertmark-Ried-Vugalter 2015]

Let $d \geq 2$. Let f be a weak solution of the Cauchy problem

$$\partial_t f = Q(f,f), \qquad f|_{t=0} = f_0 \qquad \text{(hom. BE)}$$

with initial datum $0 \leq f_0 \in L \log L(\mathbb{R}^d) \cap L^1_2(\mathbb{R}^d)$. Then, for all $0 < \alpha \leq \min\left\{\frac{\log(5/3)}{\log 2}, \nu\right\}$,

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Under slightly stronger assumptions on the kernel b (bounded on $[0, 1-\delta]$ $\forall \delta > 0$) we can improve this to

Theorem 2 [Barbaroux-Hundertmark-Ried-Vugalter 2015]

For initial conditions $f_0 \geq 0$, $f_0 \in L \log L(\mathbb{R}^d) \cap L^1_m(\mathbb{R}^d)$ with an integer

$$m \geq \max\left(2, \frac{2^\nu - 1}{2 - 2^\nu}\right),$$

any weak solution of the Cauchy problem (hom. BE) belongs to the Gevrey class $G^{\frac{1}{2\nu}}(\mathbb{R}^d)$ for strictly positive times.

In particular, for $\nu \leq \log(9/5)/\log(2) \simeq 0.847996$ we have $m = 2$ and the theorem does not require anything except the physically reasonable assumptions of finite mass, energy, and entropy.
If $\log(9/5)/\log(2) < \nu < 1$ and $f_0 \in L \log L \cap L^1_2$, then we can still conclude that the solution is in $G^{\frac{\log 2}{2\log(9/5)}}$, in particular it is ultra-analytic.

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Strategy of the Proof

- By known H^∞-smoothing result we can assume $f_0 \in L^2(\mathbb{R}^d)$.
- Take growing weights $G(\eta) = e^{\beta t \langle \eta \rangle^{2\alpha}}$ and cutoff $\mathbb{1}_\Lambda(\eta) := \mathbb{1}_{|\eta| \le \Lambda}$ in Fourier space and set

$$G_\Lambda(t, \eta) := G(t, \eta)\mathbb{1}_\Lambda(\eta)$$

- Need to control the Fourier multiplier $\|G_\Lambda(t, D_v)f(t, \cdot)\|_{L^2}$ as $\Lambda \to \infty$.
- Take $\varphi(t, \cdot) := G_\Lambda^2(t, D_v)f(t, \cdot)$ as a test function in the weak formulation. After some technicalities, this yields the L^2 reformulation of the homogeneous Boltzmann equation

$$\frac{1}{2}\|G_\Lambda f\|_{L^2}^2 - \frac{1}{2}\int_0^t \left\langle f, \left(\partial_\tau G_\Lambda^2\right) f \right\rangle \, \mathrm{d}\tau = \frac{1}{2}\|\mathbb{1}_\Lambda(D_v)f_0\|_{L^2}^2 + \int_0^t \left\langle Q(f, f), G_\Lambda^2 f \right\rangle \, \mathrm{d}\tau.$$

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In this case,

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$$\partial_\tau G_\Lambda^2(\tau,\eta) = \partial_\tau e^{2\beta\tau\langle\eta\rangle^{2\alpha}}\mathbb{1}_\Lambda(\eta) = 2\beta\langle\eta\rangle^{2\alpha} G_\Lambda^2(\tau,\eta)$$

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Bound on the commutator

By Bobylev's identity

$$CE = |\langle Q(f, G_\Lambda f) - G_\Lambda Q(f,f), G_\Lambda f\rangle|$$
$$\leq \int_{\mathbb{R}^d} \mathrm{d}\eta \int_{\mathbb{S}^{d-1}} \mathrm{d}\sigma\, b\left(\frac{\eta}{|\eta|}\cdot\sigma\right) |\widehat{f}(\eta^-)|\, |\widehat{f}(\eta^+)| |G(\eta) - G(\eta^+)| G_\Lambda(\eta) |\widehat{f}(\eta)|$$

- Here $\eta^\pm = \frac{1}{2}(\eta \pm |\eta|\sigma)$
- Note $|\eta|^2 = |\eta^-|^2 + |\eta^+|^2$, and

$$0 \leq |\eta^-| \leq |\eta^+| \quad \text{and} \quad \frac{|\eta|^2}{2} \leq |\eta^+|^2 \leq |\eta|^2$$

because of the support assumption on b.
- $b\left(\frac{\eta}{|\eta|}\cdot\sigma\right)$ has a non-integrable blow up when σ points into the direction of η, i.e., when η^+ is close to η, but then $G(\eta) - G(\eta^+)$ should be small!

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Bound on $G(\eta) - G(\eta^+)$

Let $\widetilde{G}(s) := e^{\beta t(1+s)^\alpha}$, $s = |\eta|^2$, and $s_\pm := |\eta^\pm|^2$.

- Then $s = s_+ + s_-$ and

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- Recall $\frac{s}{2} \leq s_+ \leq s$, so $\quad \frac{s - s_+}{1 + s^+} = \frac{s}{1 + s_+}\left(1 - \frac{s_+}{s}\right) \leq 2\left(1 - \frac{s_+}{s}\right)$
- Subadditivity $(1 + s)^\alpha = (1 + s_- + s_+)^\alpha \leq (1 + s_-)^\alpha + (1 + s_+)^\alpha \quad \Longrightarrow \quad \widetilde{G}(s) \leq \widetilde{G}(s_+)\widetilde{G}(s_-)$

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Bound on the commutator, continued

So we get

$$CE \leq 2\alpha\beta t \int_{\mathbb{R}^d} \mathrm{d}\eta \int_{\mathbb{S}^{d-1}} \mathrm{d}\sigma\, b\left(\tfrac{\eta}{|\eta|}\cdot\sigma\right)\left(1 - \frac{|\eta^+|^2}{|\eta|^2}\right) G(\eta^-)|\widehat{f}(\eta^-)|$$
$$\times G_\Lambda(\eta^+)|\widehat{f}(\eta^+)|\, G_\Lambda(\eta)|\widehat{f}(\eta)|\, \langle\eta^+\rangle^{2\alpha}$$

- **Good news:** The term $\left(1 - \frac{|\eta^+|^2}{|\eta|^2}\right)$ kills the singularity of b.
- **Bad news:** If we want to estimate this by L^2 based norms, one of the terms has to be bounded *uniformly*.
  The term $G(\eta^-)|\widehat{f}(\eta^-)|$ is *potentially* very strongly growing.
- **Important improvement**: Improved subadditivity for $0 \leq s_- \leq s_+$:
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Bound on the commutator, continued

So we get

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- **Good news:** The term $\left(1-\frac{|\eta^+|^2}{|\eta|^2}\right)$ kills the singularity of b.
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If we had

$$G(t,\eta^-)^{\epsilon(\alpha)}|\widehat{f}(t,\eta^-)| \le M \quad \text{for all } 0 \le t \le T$$

for some maybe large constant M, then one could conclude

$$CE \le 2\alpha\beta t M \int_{\mathbb{R}^d} \mathrm{d}\eta \int_{\mathbb{S}^{d-1}} \mathrm{d}\sigma\, b\left(\tfrac{\eta}{|\eta|}\cdot\sigma\right)\left(1-\frac{|\eta^+|^2}{|\eta|^2}\right) G_\Lambda(\eta^+)|\widehat{f}(\eta^+)|\, G_\Lambda(\eta)|\widehat{f}(\eta)|\,\langle\eta^+\rangle^{2\alpha}$$
$$\lesssim \beta T M \|G_\Lambda f\|^2_{H^\alpha}.$$

By simply choosing β small enough we would conclude as before (without commutator) that

$$\|G_\Lambda(t,D_v)f(t,\cdot)\|^2_{L^2} \le \|\mathbb{1}_\Lambda(D_v)f_0\|^2_{L^2}\, e^{2Ct}$$

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The catch

- We have to assume Gevrey regularity in order to deduce Gevrey regularity!
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The way out

Main observation: We always have $|\eta^-| \leq \frac{|\eta|}{\sqrt{2}} \leq \frac{\Lambda}{\sqrt{2}}$

- when working on a ball of radius Λ in Fourier space, the uniform bound on the 'bad term' $G(\eta^-)^{\epsilon(\alpha)}|\widehat{f}(\eta^-)|$ is only needed on the ball of radius $\Lambda/\sqrt{2}$.
- if

$$\sup_{|\zeta|\leq\Lambda} G(\zeta)^{\epsilon(\alpha)}|\widehat{f}(\zeta)| \leq M, \qquad (Hyp_\Lambda(M))$$

then the Gronwall argument, with Λ replaced by $\sqrt{2}\Lambda$ yields

$$\|G_{\sqrt{2}\Lambda}(t, D_v)f(t,\cdot)\|_{L^2}^2 \leq \|\mathbb{1}_{\sqrt{2}\Lambda}(D_v)f_0\|_{L^2}^2 e^{2Ct} \leq \|f_0\|_{L^2}^2 e^{2Ct}$$

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Possible Catch

- Need to get uniform bounds from L^2 bounds in order to control the bad term in the commutation error

Possible good news

- Need to get this uniform bounds only on smaller balls, in between Λ and $\sqrt{2}\Lambda$.
- Can assume that $\widehat{f}$ is nice, at least $\widehat{f} \in \mathscr{C}_b^2$ since $f \in L_2^1$.

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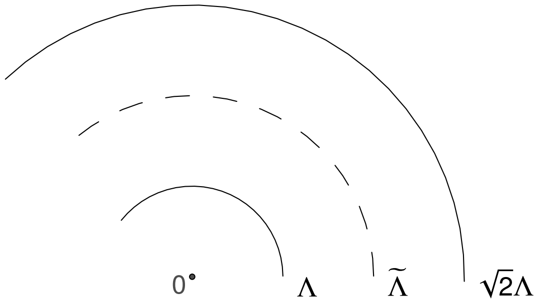

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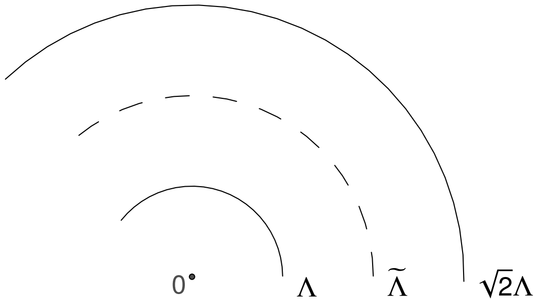
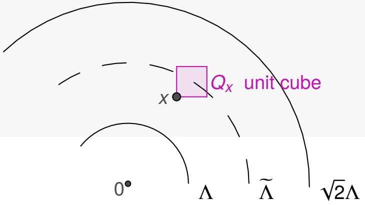


Lemma

Let $H \in \mathscr{C}^m(\mathbb{R}^n)$. Then there exists a constant $L_{m,n} < \infty$ (depending only on $m, n, \|H\|_{L^\infty(\mathbb{R}^n)}$ and, $\|D^m H\|_{L^\infty(\mathbb{R}^n)}$) such that

$$|H(x)| \le L_{m,n} \left(\int_{Q_x} |H(\xi)|^2 \, \mathrm{d}\xi \right)^{\frac{m}{2m+n}}$$



Proof for $m = 1$ easy, much trickier for $m \ge 2$!

Immediate Consequence

Since $\hat{f} \in \mathscr{C}_b^2(\mathbb{R}^d)$ and G is radially increasing, we get

$$|\hat{f}(\eta)| \le L_{2,d} \left(\int_{Q_\eta} G(\xi)^{-2} G(\xi)^2 |\hat{f}(\xi)|^2 \, \mathrm{d}\xi \right)^{\frac{2}{4+d}} \le L_{2,d} G(\eta)^{-\frac{4}{4+d}} \left(\int_{Q_\eta} G(\xi)^2 |\hat{f}(\xi)|^2 \, \mathrm{d}\xi \right)^{\frac{2}{4+d}}$$

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$$G(\eta)^{\frac{4}{4+d}} |\hat{f}(\eta)| \le L_{2,d} \|G_{\sqrt{2}\Lambda} f\|_{L^2(\mathbb{R}^d)}^{\frac{4}{4+d}} \quad \text{for all} \quad |\eta| \le \widetilde{\Lambda} = \frac{1+\sqrt{2}}{2}\Lambda$$

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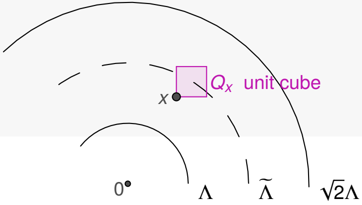

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Closing the induction

Recall $\mathrm{Hyp}_\Lambda(M)$: $\sup_{|\zeta|\le\Lambda} G(t,\zeta)^{\epsilon(\alpha)}|\widehat{f}(t,\zeta)| \le M$ for all $t \in [0,T]$

Step 0: $\mathrm{Hyp}_\Lambda(M)$ is true for some suitably chosen Λ_0

Step 1:

$$\mathrm{Hyp}_\Lambda(M) \Rightarrow \|G_{\sqrt{2}\Lambda} f\|_{L^2} \le C \quad \text{via } \textit{Gronwall}.$$

Step 2 ($L^2 \to L^\infty$ *bound*): If $\epsilon(\alpha) = 2^\alpha - 1 \le \frac{4}{4+d}$, then

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- M does not increase during the induction procedure!
- This can be accomplished by choosing β small enough at the very beginning :-)

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Thank you for you attention!

For more details, have a look at

- J.-M. Barbaroux, D. Hundertmark, T. Ried, S. Vugalter, Gevrey smoothing for weak solutions of the fully nonlinear homogeneous Boltzmann and Kac equations without cutoff for Maxwellian molecules, preprint `arXiv:1509.01444 (math.AP)`, 2015.
- J.-M. Barbaroux, D. Hundertmark, T. Ried, S. Vugalter, Strong smoothing for the non-cutoff homogeneous Boltzmann equation for Maxwellian molecules with Debye-Yukawa type interaction, preprint `arXiv:1512.05134 (math.AP)`, 2015.

Some helpful review articles

- R. Alexandre, A Review of Boltzmann Equation with Singular Kernels, *Kinetic and Related Models* **2** (2009), 551–646. `doi:10.3934/krm.2009.2.551`.
- L. Desvillettes, About the use of Fourier transform for the Boltzmann equation, *Rivista di Matematica della Università di Parma (7)* **2*** (2003), 1–99. Available at `http://www.rivmat.unipr.it/vols/2003-2s/indice.html`.
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