Karlsruhe Institute of Technology (KIT) Summer Term 2017
Institute for Analysis 02.05.2017
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# Bifurcation Theory

## Problem Sheet 2

### Problem 4 (On the Energy Method)

Find such continuous functions $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $g(0, \mu) = 0$ for all $\mu \in \mathbb{R}$ that

$$\begin{cases} -u'' = g(u, \mu) & \text{in } (0,1), \\ u(0) = u(1) = 0 \end{cases} \tag{1}$$

has the asserted properties.

**(a)** There is such $\mu_0 \in \mathbb{R}$ that (1) does not admit a nontrivial solution for $\mu \geq \mu_0$.

**(b)** There exist $\mu_0 \in \mathbb{R}$ and sequences $(u_n, \mu_n)_{n\in\mathbb{N}}, (\tilde{u}_n, \tilde{\mu}_n)_{n\in\mathbb{N}} \subseteq C^2([0,1]) \times \mathbb{R}$ of solutions of (1) which bifurcate from the trivial family at $(0, \mu_0)$ and satisfy

$$u_n, \tilde{u}_n \neq 0, \quad \mu_n < \mu_0 < \tilde{\mu}_n \quad \text{for all } n \in \mathbb{N}.$$

**(c)** We have $g \in C^2(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exist $\alpha_0 > 0$, $j \in \mathbb{N}_0$, $\lambda_0 \in \mathbb{R}$ with

$$g(-z, \mu_0) = -g(z, \mu_0), \;\; g(z, \mu_0)z > 0 \quad \text{for } 0 < |z| < \alpha_0;$$
$$\int_0^\alpha (G(\alpha, \mu_0) - G(z, \mu_0))^{-\frac{1}{2}} \, \mathrm{d}z < \infty \quad \text{for } 0 < \alpha < \alpha_0;$$
$$g_z(0, \mu_0) = \pi^2 (j+1)^2.$$

According to Theorem II.3 from the lecture, the *transversality condition* $g_{z\mu}(0, \mu_0) \neq 0$ implies that $j$-nodal solutions of (1) bifurcate from $(0, \mu_0)$ in $C^2([0,1])$. Demonstrate that bifurcation need not occur if the transversality condition is violated.

### Problem 5 (An application of the Energy Method)

Consider the boundary value problem

$$\begin{cases} u'' + \lambda(u - u^3) = 0 & \text{in } (0, \pi), \\ u(0) = u(\pi) = 0. \end{cases} \tag{2}$$

Prove that, for $k \in \mathbb{N}$, nontrivial $(k-1)$-nodal solutions of (2) bifurcate from the trivial branch at the point $\lambda_k = k^2$.

**Problem 6 (A problem without periodic Solutions)**

Let $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ with $g(0, \lambda) = 0$ $(\lambda \in \mathbb{R})$. Consider $b \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ with $b(x, \lambda) \neq 0$ for all $x, \lambda \in \mathbb{R}$.

Prove that the differential equation

$$-u'' + b(x, \lambda)\, u' = g(u, \lambda) \qquad \text{in } \mathbb{R}, \tag{3}$$

does not admit a periodic solution in $C^2(\mathbb{R})$ unless it is constant.

**Problem 7 (Variational Methods)**

Let $n \in \mathbb{N}$. We consider the $n$-dimensional bifurcation problem

$$\nabla_x I(x, \lambda) = 0 \tag{4}$$

$$\text{where } I : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}, \quad I(x, \lambda) := \frac{1}{2} x^\top A x - \lambda \sin(|x|^p)$$

for some $p \in (1, 2)$ and a symmetric, positive definite matrix $A \in \mathbb{R}^{n \times n}$.

**(a)** Show that, for $\lambda \in \mathbb{R}$, $I(\cdot, \lambda)$ is differentiable with respect to $x$ and calculate $\nabla_x I(\cdot, \lambda)$. Verify that $\nabla_x I(0, \lambda) = 0$ for all $\lambda \in \mathbb{R}$.

**(b)** Prove that, for $\lambda \in \mathbb{R}$, $I(\,\cdot\,, \lambda)$ admits a minimizer $x_\lambda \in \mathbb{R}^n$.

**(c)** Prove that $(0, 0)$ is a bifurcation point for problem (4).

**Some notes on organisation:**

**Problem Sheets**

- New problem sheets will be published on the webpage on Wednesdays. `http://www.math.kit.edu/iana2/edu/bifurcation2017s/`
- Problem sheets can be handed in for grading at the beginning of the problem classes. Grading is not compulsory.

**For questions, comments, ...**

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