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Bifurcation Theory

Solutions to Problem Sheet 3

Problem 8 (Remarks on Theorem II.3)

(a) Let $\alpha_0 > 0$ and assume that $G \in C^1((0,\alpha_0),\mathbb{R})$ with $G'(\alpha) > 0$ for $0 < \alpha < \alpha_0$. Prove that, for every $\alpha \in (0,\alpha_0)$, the following integrability condition holds:

$$\int_0^\alpha \frac{1}{\sqrt{G(\alpha) - G(z)}}\,\mathrm{d}z < \infty.$$

(b) Consider the problem

$$\begin{cases} -u'' = g(u,\lambda) & \text{in } (0,T), \\ u(0) = u(T) = 0 \end{cases} \tag{1}$$

with $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ as in Theorem II.3. Assume additionally that there exists $g_0 \in C(\mathbb{R},\mathbb{R})$ with

$$g(z,\lambda) = \lambda \cdot g_0(z) \qquad (z \in \mathbb{R}, \lambda \in \mathbb{R}).$$

Prove the following (stronger) version of Theorem II.3:

(i) For every $j \in \mathbb{N}_0$ and $\alpha \in (0,\alpha_0)$, there exists a j-nodal solution of (1) with $\|\cdot\|_\infty$-norm α.

(ii) If $g_0 \in C^1(\mathbb{R},\mathbb{R})$ and $j \in \mathbb{N}_0$, j-nodal solutions of (1) bifurcate from $(0,\lambda_j)$ in $C([0,T])$ if and only if

$$\lambda_j \cdot g_0'(0) = \left((j+1)\frac{\pi}{T}\right)^2.$$

Solution

(a) Let $\alpha \in (0, \alpha_0)$. By continuity of G' on $(0, \alpha]$, we find $\alpha_1 \in (0, \alpha)$ with

$$G'(\xi) > \frac{G'(\alpha)}{2} > 0 \qquad \text{for } \alpha_1 < \xi < \alpha,$$

and the mean value theorem yields $\xi_z \in (z, \alpha)$ with $G(\alpha) - G(z) = G'(\xi_z)(\alpha - z)$, hence

$$G(\alpha) - G(z) = G'(\xi_z)(\alpha - z) > \frac{G'(\alpha)}{2}(\alpha - z) \qquad \text{for } \alpha_1 < z < \alpha. \qquad (\diamondsuit)$$

Furthermore, $G' > 0$ on $(0, \alpha_0)$ implies that G is strictly increasing, hence in particular

$$G(\alpha) - G(z) > G(\alpha) - G(\alpha_1) > 0 \qquad \text{for } 0 < z < \alpha_1. \qquad (\heartsuit)$$

We now estimate the improper Riemann integral

$$\begin{aligned}\int_0^\alpha \frac{1}{\sqrt{G(\alpha) - G(z)}}\,\mathrm{d}z &= \int_0^{\alpha_1} \frac{1}{\sqrt{G(\alpha) - G(z)}}\,\mathrm{d}z + \int_{\alpha_1}^{\alpha} \frac{1}{\sqrt{G(\alpha) - G(z)}}\,\mathrm{d}z \\ &\overset{(\diamondsuit),(\heartsuit)}{\leq} \int_0^{\alpha_1} \frac{1}{\sqrt{G(\alpha) - G(\alpha_1)}}\,\mathrm{d}z + \int_{\alpha_1}^{\alpha} \frac{1}{\sqrt{\frac{G'(\alpha)}{2}(\alpha - z)}}\,\mathrm{d}z \\ &= \frac{\alpha_1}{\sqrt{G(\alpha) - G(\alpha_1)}} + \sqrt{\frac{2}{G'(\alpha)}} \cdot 2\sqrt{\alpha - \alpha_1} < \infty.\end{aligned}$$

This closes the proof. □

Remark:
This result shows that the integrability condition of Theorem II.3 is always satisfied.

(b) We let $g_0 \in C(\mathbb{R}, \mathbb{R})$, $T > 0$, $\alpha_0 > 0$ and $j \in \mathbb{N}_0$. Moreover, $G_0(z) := \int_0^z g_0(s)\,\mathrm{d}s$ for $z \in \mathbb{R}$. We aim to prove the following stronger version of Theorem II.3:

Corollary. Assume that $g_0(-z) = -g_0(z)$ and $g_0(z)z > 0$ hold for for $0 < |z| \leq \alpha_0$, and consider $\lambda > 0$. Then we have:

(i) For all $\alpha \in (0, \alpha_0)$ the boundary value problem

$$\begin{cases} -u'' = \lambda\, g_0(u) & \text{in } (0, T), \\ u(0) = u(T) = 0 \end{cases}$$

admits, for some $\lambda = \lambda_j(\alpha) > 0$, a periodic j-nodal solution $u \in C^2([0, T])$ with periodicity $\frac{2T}{j+1}$ and which is semi-explicitly given by

$$\sqrt{2\lambda} \cdot x = \int_0^{u(x)} \frac{1}{\sqrt{G_0(\alpha) - G_0(z)}}\,\mathrm{d}z, \qquad 0 \leq x \leq \frac{T}{2(j+1)}.$$

(ii) If $g_0 \in C^1(\mathbb{R}, \mathbb{R})$, j-nodal solutions bifurcate from $(0, \lambda_j)$ in $C([0, T])$ if and only if

$$\lambda_j \cdot g_0'(0) = \left((j+1)\frac{\pi}{T}\right)^2.$$

Proof. As g_0 is continuous, we have that $G_0 \in C^1(\mathbb{R}, \mathbb{R})$ with $G_0'(\alpha) = g_0(\alpha) > 0$ for $0 < \alpha < \alpha_0$ by assumption on g_0. Hence, part (a) yields that

$$\int_0^\alpha \frac{1}{\sqrt{G_0(\alpha) - G_0(z)}}\,\mathrm{d}z < \infty \qquad \text{for all } \alpha \in (0, \alpha_0).$$

We now apply Theorem II.3 with $g(z, \lambda) = \lambda \cdot g_0(z)$ and $G(z, \lambda) = \lambda \cdot G_0(z)$ for $\lambda > 0$. It states that a j-nodal solution with norm $\alpha \in (0, \alpha_0)$ exists under the condition

$$\frac{T}{j+1} = \int_0^\alpha \frac{1}{\sqrt{\frac{1}{2}(G(\alpha,\lambda) - G(z,\lambda))}}\,\mathrm{d}z = \sqrt{\frac{2}{\lambda}} \int_0^\alpha \frac{1}{\sqrt{G_0(\alpha) - G_0(z)}}\,\mathrm{d}z,$$

which can be satisfied by choosing

$$\lambda = \lambda_j(\alpha) := 2\left(\frac{j+1}{T}\int_0^\alpha \frac{1}{\sqrt{G_0(\alpha) - G_0(z)}}\,\mathrm{d}z\right)^2.$$

The semi-explicit formula is another direct consequence of Theorem II.3 as presented in the lecture.

Now, let g_0 (and hence g) be continuously differentiable. If bifurcation of j-nodal solutions occurs at $(0, \lambda_j)$ for some $\lambda_j > 0$, Theorem II.3 implies

$$\left((j+1)\frac{\pi}{T}\right)^2 = g_z(0, \lambda_j) = \lambda_j \cdot g_0'(0).$$

Conversely, if that identity holds, then in particular $g_0'(0) > 0$, hence $g_{z\lambda}(0, \lambda_j) = g_0'(0) \neq 0$, and Theorem II.3 states that bifurcation of j-nodal solutions occurs at $(0, \lambda_j)$.

□

Problem 9 (The direction of bifurcation)

Let $T > 0$ and, for $\lambda > 0$ and $\omega \in C([0,T],\mathbb{R})$, $\omega \geq 0$, consider the boundary value problem

$$\begin{cases} -u'' = \lambda\,\omega(x)\sin(u) & \text{in } (0,T), \\ u(0) = u(T) = 0. \end{cases} \tag{2}$$

Find $\lambda_j \in (0,\infty)$, $j \in \mathbb{N}_0$, with $\lambda_j \nearrow \infty$ and with the following property: If $u \in C^2([0,T])$ is a j-nodal solution of (2), then necessarily $\lambda \geq \lambda_j$.

Hint: Use the Sturm Comparison Theorem as introduced in the problem class.

Solution

First, let us note that for $\omega \equiv 0$, problem (2) admits only the trivial solution. We hence assume $\omega \not\equiv 0$.

Let $(u,\lambda) \in C^2([0,T]) \times (0,\infty)$ be a solution of problem (2) where u is j-nodal. We thus find an interval $(\alpha,\beta) \subseteq (0,T)$ with the following properties:

$$u(\alpha) = u(\beta) = 0, \quad u(x) \neq 0 \text{ for all } x \in (\alpha,\beta), \quad 0 < \beta - \alpha \leq \frac{T}{j+1}.$$

We now define $\tilde{u} : \mathbb{R} \to \mathbb{R}$, $\tilde{u}(x) := \sin\left(\pi \cdot \frac{x-\alpha}{\beta-\alpha}\right)$ and let $\tilde{\lambda} := \left(\frac{\pi}{\beta-\alpha}\right)^2$ and have

$$\begin{cases} u'' + \lambda\,\omega(x)\psi(u)\,u = 0 & \text{on } (\alpha,\beta), \\ u(\alpha) = 0, u(\beta) = 0, \end{cases} \qquad \begin{cases} \tilde{u}'' + \tilde{\lambda}\,\tilde{u} = 0 & \text{on } (\alpha,\beta), \\ \tilde{u}(\alpha) = 0, \tilde{u}(\beta) = 0. \end{cases} \tag{$\spadesuit$}$$

with the smooth function $\psi : \mathbb{R} \to \mathbb{R}$, $\psi(z) := \begin{cases} \frac{\sin(z)}{z} & z \neq 0, \\ 1 & z = 0, \end{cases}$ satisfying $\psi \leq 1$ on $\mathbb{R}$.

Assume for contradiction that $\lambda < \lambda_j := \frac{1}{\|\omega\|_\infty} \cdot \left(\frac{\pi(j+1)}{T}\right)^2$. This implies for all $x \in (\alpha,\beta)$

$$\lambda\,\omega(x)\psi(u(x)) \leq \lambda\ \|\omega\|_\infty < \lambda_j\ \|\omega\|_\infty = \left(\frac{\pi(j+1)}{T}\right)^2 \leq \left(\frac{\pi}{\beta-\alpha}\right)^2 = \tilde{\lambda}.$$

We recall that α and β are successive zeros of both u and $\tilde{u}$. Hence, by the Sturm Comparison Theorem applied to ($\spadesuit$), there exists $x_0 \in (\alpha,\beta)$ with $\tilde{u}(x_0) = 0$, which is the contradiction we were heading for. $\square$

We conclude that

$$\lambda \geq \lambda_j \qquad \text{where} \quad \lambda_j = \frac{1}{\|\omega\|_\infty} \cdot \left(\frac{\pi(j+1)}{T}\right)^2,$$

and remark that for $\omega \equiv 1$ (in the case of the pendulum equation), these λ_j coincide with the bifurcation points we found in the lecture, i.e. that bifurcation directs “to the right“.

Problem 10 (Gâteaux Differentiability)

Let X and Z be Banach spaces, $U \subseteq X$ an open subset and $x \in U$. A map $F : U \to Z$ is said to be *Gâteaux differentiable* in x if there exists a continuous linear operator $A \in \mathcal{L}(X, Z)$ with the property that, for every $h \in X$,

$$\lim_{\tau \to 0} \frac{F(x + \tau h) - F(x)}{\tau} = Ah.$$

In this case, we call $A =: \mathrm{d}F(x)$ the *Gâteaux derivative* of F in x.

(a) Assume that $F : U \to Z$ is Gâteaux differentiable in every $x \in U$, and that the map

$$\mathrm{d}F : U \to \mathcal{L}(X, Z), \quad x \mapsto \mathrm{d}F(x)$$

is continuous. Prove that F is continuously Fréchet differentiable on U, and that its Fréchet derivative is given by the Gâteaux derivative, i.e. $F'(x)[h] = \mathrm{d}F(x)[h]$ holds for all $x \in U, h \in X$.

(b) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $\varphi \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$. We consider the Banach space $C(\overline{\Omega})$ endowed with the norm $\|u\|_\infty := \max_{x \in \overline{\Omega}} |u(x)|$, $u \in C(\overline{\Omega})$.

Prove that the map

$$F : C(\overline{\Omega}, \mathbb{R}) \to C(\overline{\Omega}, \mathbb{R}), \quad (F(u))(x) := \varphi(x, u(x)) \qquad (x \in \overline{\Omega})$$

is continuously Fréchet differentiable and calculate its derivative.

Solution

(a) Let $x \in U$ and $\varepsilon > 0$. By continuity of the Gâteaux derivative, we find $\delta > 0$ with $B_\delta(x) \subseteq U$ and

$$\|\mathrm{d}F(x + y) - \mathrm{d}F(x)\|_{\mathcal{L}(X,Z)} < \varepsilon \quad \text{whenever} \quad y \in X,\ \|y\|_X < \delta. \qquad (\clubsuit)$$

We consider an arbitrary element of the dual space $z' \in Z'$ and $h \in X$ with $\|h\|_X < \delta$ and (without loss of generality) so small that the following function is well-defined:

$$f : (-1, 1) \to \mathbb{R}, \quad f(\tau) := z'(F(x + \tau h)).$$

Assertion: f is differentiable with $f'(\tau) = z'(\mathrm{d}F(x + \tau h)[h])$.
Proof: For $\sigma \in \mathbb{R}$, $\sigma \to 0$, we have with $\tilde{x} := x + \tau h$

$$\begin{aligned}
&|f(\tau + \sigma) - f(\tau) - \sigma z'(\mathrm{d}F(x + \tau h)[h])| \\
&\quad = |z'(F(x + \tau h + \sigma h) - F(x + \tau h) - \sigma \mathrm{d}F(x + \tau h)[h])| \\
&\quad \le \|z'\|_{Z'} \|F(x + \tau h + \sigma h) - F(x + \tau h) - \sigma \mathrm{d}F(x + \tau h)[h]\|_Z \\
&\quad = \|z'\|_{Z'} \|F(\tilde{x} + \sigma h) - F(\tilde{x}) - \sigma \mathrm{d}F(\tilde{x})[h]\|_Z = o(\sigma)
\end{aligned}$$

by definition of Gâteaux differentiability. ▪

As a consequence, the fundamental theorem of calculus gives

$$\begin{aligned} z'(F(x+h) - F(x) - \mathrm{d}F(x)[h]) = f(1) - f(0) - f'(0) &= \int_0^1 f'(\tau) - f'(0)\,\mathrm{d}\tau \\ &= \int_0^1 z'(\mathrm{d}F(x+\tau h)[h] - \mathrm{d}F(x)[h])\,\mathrm{d}\tau. \end{aligned}$$

Since $\|\tau h\|_X \leq \|h\|_X < \delta$ for $0 \leq \tau \leq 1$, estimate (♣) yields

$$\begin{aligned} |z'(F(x+h) - F(x) - \mathrm{d}F(x)[h])| &\leq \int_0^1 \|z'\|_{Z'} \|\mathrm{d}F(x+\tau h) - \mathrm{d}F(x)\|_{\mathcal{L}(X,Z)} \|h\|_X \,\mathrm{d}\tau \\ &\leq \varepsilon \|z'\|_{Z'} \|h\|_X . \end{aligned}$$

By a consequence of the Hahn-Banach Theorem[1], we may conclude

$$\|F(x+h) - F(x) - \mathrm{d}F(x)h\|_Z \leq \varepsilon \|h\|_X$$

for all $h \in X$ with $\|h\|_X < \delta$ since $z' \in Z'$ was arbitrary. This yields $F(x+h) - F(x) - \mathrm{d}F(x)[h] = o(\|h\|_X)$; hence, F is Fréchet differentiable in every $x \in U$ with $F'(x) = \mathrm{d}F(x)$, and continuity of the Fréchet derivative is a direct consequence of continuity of the Gâteaux derivative (which was assumed). □

(b) We intend to apply the result of part (a); this requires two steps.[2]

Step 1: Gâteaux differentiability of F

Let $u, h \in C(\overline{\Omega})$. By definition of Gâteaux differentiability, we have to find a linear continuous operator $A \in \mathcal{L}(C(\overline{\Omega}), C(\overline{\Omega}))$ with

$$\lim_{\tau \to 0} \frac{\|F(u+\tau h) - F(u) - \tau A h\|_\infty}{\tau} = 0.$$

First, we prove the existence of a pointwise limit and thus fix $x \in \overline{\Omega}$. Then,

$$\begin{aligned} \lim_{\tau \to 0} \frac{(F(u+\tau h))(x) - (F(u))(x)}{\tau} &= \lim_{\tau \to 0} \frac{\varphi(x, u(x) + \tau h(x)) - \varphi(x, u(x))}{\tau} \\ &= \frac{\partial \varphi}{\partial x_{n+1}}(x, u(x)) \cdot h(x) =: \varphi_u(x, u(x)) \cdot h(x) \end{aligned}$$

which is a consequence of the chain rule and of the differentiability of φ. Hence, the only candidate for a Gâteaux derivative is $(\mathrm{d}F(u)[h])(x) = \varphi_u(x, u(x)) \cdot h(x)$.

[1] for a reference in German, cf. Werner, Funktionalanalysis (7. Auflage), Korollar III.1.6; we use that, for $z \in Z$, $\|z\|_Z = \sup\{z'(z) : z' \in Z', \|z'\|_{Z'} = 1\}$.

[2] cf. Chang, Methods in Nonlinear Analysis, p. 4

In a second step, we have to show that this limit is uniform (i.e. that we have convergence in the Banach space $C(\overline{\Omega})$). Let $\varepsilon > 0$. Since φ_u is uniformly continuous on the compact set $\overline{\Omega} \times [-M, M]$, $M := \|u\|_\infty + \|h\|_\infty$, we find $\delta > 0$ with

$$\begin{aligned}&\forall\, x \in \overline{\Omega} \quad \forall\, z_1, z_2 \in [-M, M] :\\ &|z_1 - z_2| < \delta \Rightarrow |\varphi_u(x, z_1) - \varphi_u(x, z_2)| < \frac{\varepsilon}{\max\{1, \|h\|_\infty\}}.\end{aligned}$$

So for all $\tilde{\tau} \in [-1, 1]$ with $|\tilde{\tau}|\, \|h\|_\infty < \delta$, we have

$$\forall\, x \in \overline{\Omega} : \qquad |\varphi_u(x, u(x) + \tilde{\tau} h(x)) - \varphi_u(x, u(x))| < \frac{\varepsilon}{\max\{1, \|h\|_\infty\}}$$

and find for $|\tau| \in (0, 1)$, using the fundamental theorem of calculus,

$$\begin{aligned}&\max_{x\in\overline{\Omega}} \left| \frac{\varphi(x, u(x) + \tau h(x)) - \varphi(x, u(x)) - \tau \varphi_u(x, u(x)) h(x)}{\tau} \right|\\ &= \max_{x\in\overline{\Omega}} \left| \frac{1}{\tau} \left(\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\sigma} \Big[\varphi(x, u(x) + \sigma\tau h(x)) \Big] - \tau \varphi_u(x, u(x)) h(x) \, \mathrm{d}\sigma \right) \right|\\ &= \max_{x\in\overline{\Omega}} \left| \int_0^1 \varphi_u(x, u(x) + \sigma\tau h(x)) h(x) - \varphi_u(x, u(x)) h(x) \, \mathrm{d}\sigma \right|\\ &\leq \max_{x\in\overline{\Omega}} \int_0^1 \big| \varphi_u(x, u(x) + \sigma\tau h(x)) - \varphi_u(x, u(x)) \big| |h(x)| \, \mathrm{d}\sigma\\ &< \max_{x\in\overline{\Omega}} \int_0^1 \frac{\varepsilon}{\max\{1, \|h\|_\infty\}} \|h\|_\infty \, \mathrm{d}\sigma = \frac{\varepsilon}{\max\{1, \|h\|_\infty\}} \|h\|_\infty \leq \varepsilon.\end{aligned}$$

As the partial derivative φ_u is continuous, we define

$$A : C(\overline{\Omega}) \to C(\overline{\Omega}), \qquad (Ah)(x) := \varphi_u(x, u(x)) \cdot h(x), \;\; x \in \Omega$$

the estimate above shows $\lim_{\tau\to 0} \frac{\|F(u+\tau h) - F(u) - \tau Ah\|_\infty}{\tau} = 0$. Furthermore, for fixed $u \in C(\overline{\Omega})$, A is linear and continuous, and we conclude that F is Gâteaux differentiable with derivative $\mathrm{d}F(u) = A$.

Step 2: Continuity of the Gâteaux derivative of F

We consider functions $u_n, u \in C(\overline{\Omega})$, $n \in \mathbb{N}$, with $\|u_n - u\|_\infty \to 0$ as $n \to \infty$. We intend to prove that $\mathrm{d}F(u_n) \to \mathrm{d}F(u)$ as $n \to \infty$ in $\mathcal{L}(C(\overline{\Omega}), C(\overline{\Omega}))$, i.e.

$$\sup_{h\in C(\overline{\Omega}),\, \|h\|_\infty = 1} \|\mathrm{d}F(u_n)[h] - \mathrm{d}F(u)[h]\|_\infty \to 0.$$

We let $\varepsilon > 0$ and, using uniform continuity of φ_u in a similar way as in the previous part, we find $\delta > 0$ with

$$\begin{aligned}&\forall\, x \in \overline{\Omega} \quad \forall\, z_1, z_2 \in [-M_1, M_1] :\\ &|z_1 - z_2| < \delta \Rightarrow |\varphi_u(x, z_1) - \varphi_u(x, z_2)| < \varepsilon\end{aligned}$$

where $M_1 := \|u\|_\infty + 1$. As $u_n \to u$ uniformly on $\overline{\Omega}$, we also find $n_0 \in \mathbb{N}$ with $\|u_n\|_\infty \le M_1$ and $|u_n(x) - u(x)| < \delta$ for all $n \ge n_0$ and $x \in \overline{\Omega}$. We estimate for $n \ge n_0$

$$\begin{aligned}
&\sup_{h \in C(\overline{\Omega}),\, \|h\|_\infty = 1} \|\mathrm{d}F(u_n)[h] - \mathrm{d}F(u)[h]\|_\infty \\
&\quad \le \sup_{h \in C(\overline{\Omega}),\, \|h\|_\infty = 1} \left(\max_{x \in \overline{\Omega}} |\varphi_u(x, u_n(x)) - \varphi_u(x, u(x))| \cdot \|h\|_\infty \right) \\
&\quad = \max_{x \in \overline{\Omega}} |\varphi_u(x, u_n(x)) - \varphi_u(x, u(x))| < \varepsilon,
\end{aligned}$$

which shows continuity of the Gâteaux derivative. By part (a), we conclude that F is continuously Fréchet differentiable with $F'(u)[h] = \mathrm{d}F(u)[h]$. $\square$