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Bifurcation Theory

Solutions to Problem Sheet 5

Problem 14 (Another application of the Implicit Function Theorem)

Let $n \in \mathbb{N}$. Consider a smooth bounded domain $\Omega \subseteq \mathbb{R}^n$ and, for $\varepsilon > 0$, the boundary value problem

$$(\clubsuit)_\varepsilon \quad \begin{cases} -\Delta u = u^3 & \text{in } \Omega, \\ u \equiv \varepsilon & \text{on } \partial\Omega. \end{cases}$$

Prove that there exists $\varepsilon_0 > 0$ with the property that problem $(\clubsuit)_\varepsilon$ admits a classical solution $u_\varepsilon \in C^2(\overline{\Omega})$ for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

Solution

We fix an arbitrary $\alpha \in (0,1)$ and show that there exist solutions $u_\varepsilon \in C^{2,\alpha}(\overline{\Omega})$ to $(\clubsuit)_\varepsilon$ for sufficiently small $|\varepsilon|$.

We intend to exploit the homeomorphism

$$(-\Delta)^{-1} : C^{0,\alpha}(\overline{\Omega}) \to C^{2,\alpha}_{\text{Dirichlet}}(\overline{\Omega}) \qquad \text{where} \quad C^{2,\alpha}_{\text{Dirichlet}}(\overline{\Omega}) := \{u \in C^{2,\alpha}(\overline{\Omega}) : \ u\big|_{\partial\Omega} \equiv 0\}$$

as introduced in the problem class. To this end, we consider the shifted problems

$$(\clubsuit)'_\varepsilon \quad \begin{cases} -\Delta v = (v+\varepsilon)^3 & \text{in } \Omega, \\ v \equiv 0 & \text{on } \partial\Omega \end{cases}$$

and note that, for $u, v : \overline{\Omega} \to \mathbb{R}$ with $v(x) = u(x) - \varepsilon$ for all $x \in \overline{\Omega}$, we have the equivalence

$$u \in C^{2,\alpha}(\overline{\Omega}), \quad u \text{ solves } (\clubsuit)_\varepsilon \qquad \Leftrightarrow \qquad v \in C^{2,\alpha}(\overline{\Omega}), \quad v \text{ solves } (\clubsuit)'_\varepsilon.$$

Further, for $v \in C^{2,\alpha}(\overline{\Omega})$, $(v+\varepsilon)^3 \in C^2(\overline{\Omega}) \subseteq C^{0,\alpha}(\overline{\Omega})$; hence we can define

$$F : C^{2,\alpha}_{\text{Dirichlet}}(\overline{\Omega}) \times \mathbb{R} \to C^{2,\alpha}_{\text{Dirichlet}}(\overline{\Omega}), \qquad F(v,\varepsilon) := v - (-\Delta)^{-1}\left[(v+\varepsilon)^3\right].$$

Again, for $(v,\varepsilon) \in C^{2,\alpha}_{\text{Dirichlet}}(\overline{\Omega}) \times \mathbb{R}$ and $u := v + \varepsilon$

$$F(v,\varepsilon) = 0 \quad \Leftrightarrow \quad v \text{ solves } (\clubsuit)'_\varepsilon \quad \Leftrightarrow \quad u \text{ solves } (\clubsuit)_\varepsilon.$$

In particular, $F(0,0) = 0$, and the assertion is proved if we find $\varepsilon_0 > 0$ with the property that, for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, there exists $v_\varepsilon \in C^{2,\alpha}_{\text{Dirichlet}}(\overline{\Omega})$ with $F(v_\varepsilon, \varepsilon) = 0$. This will be done using the Implicit Function Theorem.

For the estimates proving continuous differentiability, we need an algebraic property of the Hölder norms.

Lemma. For $a, b \in C^{0,\alpha}(\overline{\Omega})$, we have that $ab \in C^{0,\alpha}(\overline{\Omega})$ with

$$\|ab\|_{C^{0,\alpha}(\overline{\Omega})} \leq 3\,\|a\|_{C^{0,\alpha}(\overline{\Omega})}\,\|b\|_{C^{0,\alpha}(\overline{\Omega})}. \qquad (\diamondsuit)$$

Proof of Lemma: Let $a, b \in C^{0,\alpha}(\overline{\Omega})$. First, we note that ab is a continuous function. We now estimate the norms, recalling that

$$\|a\|_{C^{0,\alpha}(\overline{\Omega})} = \|a\|_\infty + [a]_{0,\alpha} = \sup_{x\in\overline{\Omega}} |a(x)| + \sup_{x\neq y\in\overline{\Omega}} \frac{|a(x)-a(y)|}{|x-y|^\alpha}.$$

We have $\|ab\|_\infty \leq \|a\|_\infty \|b\|_\infty \leq \|a\|_{C^{0,\alpha}(\overline{\Omega})} \|b\|_{C^{0,\alpha}(\overline{\Omega})}$ and

$$\begin{aligned}
[ab]_{0,\alpha} &= \sup_{x\neq y\in\overline{\Omega}} \frac{|a(x)b(x)-a(y)b(y)|}{|x-y|^\alpha} \leq \sup_{x\neq y\in\overline{\Omega}} \frac{|a(x)||b(x)-b(y)| + |a(x)-a(y)||b(y)|}{|x-y|^\alpha} \\
&\leq \|a\|_\infty \sup_{x\neq y\in\overline{\Omega}} \frac{|b(x)-b(y)|}{|x-y|^\alpha} + \|b\|_\infty \sup_{x\neq y\in\overline{\Omega}} \frac{|a(x)-a(y)|}{|x-y|^\alpha} \\
&= \|a\|_\infty [b]_{0,\alpha} + \|b\|_\infty [a]_{0,\alpha} \leq 2\,\|a\|_{C^{0,\alpha}(\overline{\Omega})}\,\|b\|_{C^{0,\alpha}(\overline{\Omega})},
\end{aligned}$$

and summing up both estimates, the lemma is proved. ■

Moreover, we fix a constant $C_2 > 0$ with $\|w\|_{C^{0,\alpha}(\overline{\Omega})} \leq C_2\,\|w\|_{C^{2,\alpha}(\overline{\Omega})}$ for all $w \in C^{2,\alpha}(\overline{\Omega})$.

Now, we proceed in three steps.

(i) *Fréchet differentiability of F w.r.t. $v \in C^{2,\alpha}(\overline{\Omega})$.*

Let $\varepsilon \in \mathbb{R}$ and $v, h \in C^{2,\alpha}_{\text{Dirichlet}}(\overline{\Omega})$. We will prove that F is Fréchet differentiable w.r.t. v with

$$D_v F(v,\varepsilon)[h] = h - 3\,(-\Delta)^{-1}[(v+\varepsilon)^2 h].$$

We let $\beta := \|(-\Delta)^{-1}\|_{\mathcal{L}(C^{0,\alpha}(\overline{\Omega}),C^{2,\alpha}(\overline{\Omega}))}$ and estimate as follows using $(\diamondsuit)$

$$\begin{aligned}
&\left\|F(v+h,\varepsilon) - F(v,\varepsilon) - \left(h - 3\,(-\Delta)^{-1}[(v+\varepsilon)^2 h]\right)\right\|_{C^{2,\alpha}(\overline{\Omega})} \\
&\quad= \left\|(-\Delta)^{-1}\left[(v+h+\varepsilon)^3 - (v+\varepsilon)^3 - 3\,(v+\varepsilon)^2 h\right]\right\|_{C^{2,\alpha}(\overline{\Omega})} \\
&\quad\leq \beta\,\left\|3(v+\varepsilon)h^2 + h^3\right\|_{C^{0,\alpha}(\overline{\Omega})} \leq \beta\left(3\,\left\|(v+\varepsilon)h^2\right\|_{C^{0,\alpha}(\overline{\Omega})} + \left\|h^3\right\|_{C^{0,\alpha}(\overline{\Omega})}\right) \\
&\quad\leq 9C_2^2\beta\,\|h\|^2_{C^{2,\alpha}(\overline{\Omega})} \cdot \left(3\,\|v+\varepsilon\|_{C^{0,\alpha}(\overline{\Omega})} + \|h\|_{C^{0,\alpha}(\overline{\Omega})}\right) = o(\|h\|_{C^{2,\alpha}(\overline{\Omega})}),
\end{aligned}$$

which proves differentiability.

(ii) *Continuous Fréchet differentiability of F w.r.t. $v \in C^{2,\alpha}(\overline{\Omega})$.*

To show continuity of the derivative, we consider $v, v_n = v + q_n \in C^{2,\alpha}_{\text{Dirichlet}}(\overline{\Omega})$ with $v_n \to v$ as $n \to \infty$. Then $q_n \to 0$ in $C^{2,\alpha}(\overline{\Omega})$. Similar to part (i), we estimate using $(\diamondsuit)$

$$\begin{aligned}
&\|D_vF(v,\varepsilon) - D_vF(v_n,\varepsilon)\|_{\mathcal{L}(C^{2,\alpha}(\overline{\Omega}),C^{2,\alpha}(\overline{\Omega}))}\\
&= \sup_{w\in C^{2,\alpha}(\overline{\Omega}),\, \|w\|_{C^{2,\alpha}(\overline{\Omega})}=1} \left\|3\,(-\Delta)^{-1}[(v+\varepsilon)^2 w] - 3\,(-\Delta)^{-1}[(v+q_n+\varepsilon)^2 w]\right\|_{C^{2,\alpha}(\overline{\Omega})}\\
&\le 3\beta \sup_{w\in C^{2,\alpha}(\overline{\Omega}),\, \|w\|_{C^{2,\alpha}(\overline{\Omega})}=1} \left\|(v+\varepsilon)^2 w - (v+q_n+\varepsilon)^2 w\right\|_{C^{0,\alpha}(\overline{\Omega})}\\
&\le 3\beta \sup_{w\in C^{2,\alpha}(\overline{\Omega}),\, \|w\|_{C^{2,\alpha}(\overline{\Omega})}=1} \left\|2q_n(v+\varepsilon)w + q_n^2 w\right\|_{C^{0,\alpha}(\overline{\Omega})}\\
&\le 27\beta \sup_{w\in C^{2,\alpha}(\overline{\Omega}),\, \|w\|_{C^{2,\alpha}(\overline{\Omega})}=1} \|q_n\|_{C^{0,\alpha}(\overline{\Omega})} \|w\|_{C^{0,\alpha}(\overline{\Omega})} \Big(2\,\|v+\varepsilon\|_{C^{0,\alpha}(\overline{\Omega})} + \|q_n\|_{C^{0,\alpha}(\overline{\Omega})}\Big)\\
&\le 27C_2^2\beta\,\|q_n\|_{C^{2,\alpha}(\overline{\Omega})} \Big(2\,\|v+\varepsilon\|_{C^{0,\alpha}(\overline{\Omega})} + C_2\,\|q_n\|_{C^{2,\alpha}(\overline{\Omega})}\Big) \to 0 \qquad \text{as } n\to\infty.
\end{aligned}$$

This proves continuity of the Fréchet derivative.

(iii) *Application of the Implicit Function Theorem.*

By part (i), we have for $(v,\varepsilon) \in C^{2,\alpha}_{\text{Dirichlet}}(\overline{\Omega}) \times \mathbb{R}$ and $w \in C^{2,\alpha}_{\text{Dirichlet}}(\overline{\Omega})$

$$D_vF(v,\varepsilon)[w] = w - p\,(-\Delta)^{-1}[(v+\varepsilon)^2 w]$$

and in particular

$$D_vF(0,0)[w] = w, \quad \text{hence} \quad D_vF(0,0) = I_{C^{2,\alpha}_{\text{Dirichlet}}(\overline{\Omega})}.$$

Then condition (i) in the IFT, Theorem III.7, holds because the identity is a linear homeomorphism of $C^{2,\alpha}_{\text{Dirichlet}}(\overline{\Omega})$, and (ii) holds since F is continuously differentiable w.r.t. $v \in C^{2,\alpha}_{\text{Dirichlet}}(\overline{\Omega})$ (as shown in part (ii)).

Hence, there exist an open neighborhood $J \subseteq \mathbb{R}$ of 0 and a continuous function $\hat{v} : J \to C^{2,\alpha}_{\text{Dirichlet}}(\overline{\Omega})$ with

$$F(\hat{v}(\varepsilon),\varepsilon) = 0 \qquad \text{for all } \varepsilon \in J.$$

Choosing $\varepsilon_0 > 0$ with $(-\varepsilon_0,\varepsilon_0) \subseteq J$ and defining $v_\varepsilon := \hat{v}(\varepsilon)$, the proof is complete. □

Problem 15 (Choosing the proper function space)

Let $n \in \mathbb{N}$, and let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary. In the previous exercise class, we introduced such pairs of spaces X, Z containing functions on Ω with values in $\mathbb{R}$ that the mapping

$$(\spadesuit) \qquad (-\Delta)^{-1}: \ X \to Z, \ f \mapsto u \qquad \text{where} \quad \begin{cases} -\Delta u = f & \text{in } \Omega, \\ u \equiv 0 & \text{on } \partial\Omega \end{cases}$$

is a linear homeomorphism.

(a) Show that this cannot hold for $Z := H_0^3(\Omega)$ and $X := H^1(\Omega)$.

(b) Construct a counterexample in the case $n = 1$, Ω being a bounded open interval, which shows that ($\spadesuit$) is in general not well-defined when choosing $X := Z := C_0^\infty(\Omega)$.

Solution

(a) Quoting the results from the exercise class, we know that ($\spadesuit$) is a linear homeomorphism for the choice $X := H^1(\Omega)$, $Z := H_0^1(\Omega) \cap H^3(\Omega)$. Since $H_0^1(\Omega) \cap H^3(\Omega) \supsetneqq H_0^3(\Omega)$, we infer that the mapping given in part (a) cannot be well-defined.

We will now give a direct proof of this result. To this end, we assume for contradiction that ($\spadesuit$) with $Z := H_0^3(\Omega)$ and $X := H^1(\Omega)$ is a linear homeomorphism. We choose[1]

$$f \in H^1(\Omega) \setminus H_0^1(\Omega)$$

and hence find $u \in H_0^3(\Omega)$ with $-\Delta u = f$ in Ω. By definition of $H_0^3(\Omega)$, there exist $u_n \in C_0^\infty(\Omega)$, $n \in \mathbb{N}$, with $\|u_n - u\|_{H^3(\Omega)} \to 0$ as $n \to \infty$. Setting $f_n := -\Delta u_n$ (with classical derivatives!), we infer $f_n \in C_0^\infty(\Omega)$ and

$$\|f_n - f\|_{H^1(\Omega)} = \|-\Delta u_n + \Delta u\|_{H^1(\Omega)} \leq c \, \|u_n - u\|_{H^3(\Omega)} \to 0$$

as $n \to \infty$ for some constant $c > 0$ depending on the precise definition of the Sobolev norms. Again by definition of $H_0^1(\Omega)$, this implies $f \in H_0^1(\Omega)$, a contradiction.

(b) Let $n = 1$, (w.l.o.g.) $\Omega := (-2, 2)$ and consider[2] $X := Z := C_0^\infty(\Omega)$ as well as a smooth function $f \in C_0^\infty((-2, 2))$ with $f \geq 0$, $f(x) = 0$ if $1 < |x| < 2$ and $f(0) > 0$.

We assume for contradiction that there exists $u := (-\Delta)^{-1} f \in C_0^\infty((-2, 2))$. In particular, $u(-2) = u'(-2) = 0$ due to compact support, and we find by integration:

$$\begin{aligned} u(2) &= u(-2) + \int_{-2}^{2} u'(t) \, \mathrm{d}t = u(-2) + \int_{-2}^{2} \left(u'(-2) + \int_{-2}^{t} u''(s) \, \mathrm{d}s \right) \, \mathrm{d}t \\ &= - \int_{-2}^{2} \int_{-2}^{t} f(s) \, \mathrm{d}s \mathrm{d}t < 0 \end{aligned}$$

by assumption on f, which contradicts $u \in C_0^\infty((-2, 2))$. □

[1] Since Ω is bounded, we can choose $f \equiv 1$. Then $f \in H^1(\Omega)$, even $f \in C^1(\overline{\Omega})$. However, since $f\big|_{\partial\Omega} \not\equiv 0$, we conclude $f \notin H_0^1(\Omega)$. Note that the latter is a consequence of the (nontrivial) *trace theorem*, cf. Theorems 1 and 2 of chapter 5.5, L.C.Evans, *Partial Differential Equations*, Volume 19 of Graduate Studies in Mathematics, AMS 2002.

[2] Note that we do not have a norm on these spaces, so we do not assess continuity.