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Bifurcation Theory

Solutions to Problem Sheet 7

Problem 19 (Bifurcation Formulae in finite dimensions)

As in Problem 17 (b), let $A \in \mathbb{R}^{n\times n}$ be symmetric, and $J \in \mathbb{R}^{n\times n}$. We study

$$Ax = \lambda x + |x|^2 Jx \qquad (x \in \mathbb{R}^n, \lambda \in \mathbb{R}). \tag{1}$$

We have seen that, if λ_0 is a simple eigenvalue of A, a continuously differentiable curve of nontrivial solutions $(\hat{x}(s), \hat{\lambda}(s))_{-\delta<s<\delta}$ of problem (1) bifurcates from $(0, \lambda_0)$.

Show that $\hat{\lambda}'(0) = 0$ and calculate $\hat{\lambda}''(0)$.

Solution

First, we recall some notation. We let

$$F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \quad F(x, \lambda) := Ax - \lambda x - |x|^2 Jx$$

and denote by $(\mu_j, \psi_j) \in \mathbb{R} \times \mathbb{R}^n$, $j = 1, ..., n$, the eigenpairs of the (symmetric!) matrix A where, as in Problem 17 (b), $\mu_1 = \lambda_0 \neq \mu_j$ for $j = 2, ..., n$ and the eigenvectors are orthonormal. We note that $F \in C^\infty(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ and calculate for $x, v_1, v_2, v_3 \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$:

$$\begin{aligned}
F_x(x, \lambda)[v_1] &= (A - \lambda I)v_1 - |x|^2 Jv_1 - 2\langle x, v_1\rangle Jx,\\
F_{xx}(x, \lambda)[v_1, v_2] &= -2\langle x, v_2\rangle Jv_1 - 2\langle x, v_1\rangle Jv_2 - 2\langle v_1, v_2\rangle Jx,\\
F_{xxx}(x, \lambda)[v_1, v_2, v_3] &= -2\langle v_3, v_2\rangle Jv_1 - 2\langle v_3, v_1\rangle Jv_2 - 2\langle v_1, v_2\rangle Jv_3,\\
F_{x\lambda}(x, \lambda)[v_1] &= -v_1.
\end{aligned}$$

In the bifurcation point $(0, \lambda_0)$, we have seen that $\ker F_x(0, \lambda_0) = (\operatorname{ran} F_x(0, \lambda_0))^\perp = \operatorname{span} \psi_1$.

In the context of the bifurcation formulae, we choose $\varphi := \psi_1$ and $\varphi' : \mathbb{R}^n \to \mathbb{R},\ x \mapsto \langle x, \psi_1\rangle$. We have $F_{xx}(0, \lambda_0) \equiv 0$, which immediately yields

$$\begin{aligned}
&\hat{\lambda}'(0) = 0,\\
&\hat{\lambda}''(0) = -\frac{1}{3}\frac{\varphi'(F_{xxx}(0, \lambda_0)[\varphi, \varphi, \varphi])}{\varphi'(F_{x\lambda}(0, \lambda_0)[\varphi])} = -\frac{1}{3}\frac{\langle -6J\psi_1, \psi_1\rangle}{\langle -\psi_1, \psi_1\rangle} = -2\langle J\psi_1, \psi_1\rangle.
\end{aligned}$$

□

Problem 20 (Bifurcation Formulae for Problem 18)

Problem 18 shows that, for $n \geq 2$, the ODE

$$-u'' = \lambda u + u^n \qquad \text{on } \mathbb{R}, \tag{2}$$

admits a continuously differentiable curve $(\hat{u}(s), \hat{\lambda}(s))_{-\delta<s<\delta} \subseteq C^2(\mathbb{R}) \times \mathbb{R}$ of nontrivial 2π-periodic solutions bifurcating from $(0,0)$.

For $n = 2$ and $n = 3$, sketch the bifurcation diagram near $(0,0)$ by calculating $\hat{\lambda}'(0)$ and, if the first derivative vanishes, $\hat{\lambda}''(0)$.

Here and in the next problem, you do not need to prove higher-order Fréchet differentiability.

Solution

Let us first note that the choice $g(x,z,\lambda) = z^n$, $n = 2$ or $n = 3$, satisfies the assumptions of Problem 18. Again, we let $F : C^2_{\text{per}}(\mathbb{R}) \times \mathbb{R} \to C_{\text{per}}(\mathbb{R}),\ F(u,\lambda) := u'' + \lambda u + u^n$ and recall that $(0,0)$ is a bifurcation point in the sense of Theorem IV.2; moreover,

$$\ker\left(F_u(0,0)\right) = \text{span}\ \{\mathbb{1}\}, \quad \text{ran}\left(F_u(0,0)\right) = \left\{z \in C_{\text{per}}(\mathbb{R}) : \int_0^{2\pi} z(t)\,\mathrm{d}t = 0\right\}$$

and span $\{\mathbb{1}\} \oplus \left\{z \in C_{\text{per}}(\mathbb{R}) : \int_0^{2\pi} z(t)\,\mathrm{d}t = 0\right\} = C_{\text{per}}(\mathbb{R})$. In the context of the bifurcation formulae, we define

$$\varphi := \mathbb{1} \quad \text{and} \quad \varphi' : C_{\text{per}}(\mathbb{R}) \to \mathbb{R},\ \varphi'[z] := \frac{1}{2\pi}\int_0^{2\pi} z(t)\,\mathrm{d}t.$$

Then, $\varphi'[\varphi] = 1$ and ran $(F_u(0,0)) = \ker \varphi'$.

Let us first consider the case $n = 2$.

We give (without proof) for $u, v_1, v_2 \in C^2_{\text{per}}(\mathbb{R})$ and $\lambda \in \mathbb{R}$ the following Fréchet derivatives

$$F_u(u,\lambda)[v_1] = v_1'' + \lambda v_1 + 2uv_1, \quad F_{uu}(u,\lambda)[v_1,v_2] = 2v_1v_2, \quad F_{u\lambda}(u,\lambda)[v_1] = v_1.$$

We then calculate

$$\hat{\lambda}'(0) = -\frac{1}{2}\frac{\varphi'[F_{uu}(0,0)[\varphi,\varphi]]}{\varphi'[F_{u\lambda}(0,0)[\varphi]]} = -\frac{1}{2}\frac{\varphi'[2\varphi]}{\varphi'[\varphi]} = -1.$$

Let us now consider the case $n = 3$.

Again, we give without proof for $u, v_1, v_2, v_3 \in C^2_{\text{per}}(\mathbb{R})$ and $\lambda \in \mathbb{R}$

$F_u(u,\lambda)[v_1] = v_1'' + \lambda v_1 + 3u^2 v_1$, $F_{uu}(u,\lambda)[v_1,v_2] = 6uv_1v_2$, $F_{uuu}(u,\lambda)[v_1,v_2,v_3] = 6v_1v_2v_3$, $F_{u\lambda}(u,\lambda)[v_1] = v_1$.

Noting that $F_{uu}(0,0) \equiv 0$, we find $\hat{\lambda}'(0) = 0$ and

$$\hat{\lambda}''(0) = -\frac{1}{3}\frac{\varphi'(F_{uuu}(0,0)[\varphi,\varphi,\varphi])}{\varphi'(F_{u\lambda}(0,0)[\varphi])} = -\frac{1}{3}\frac{\varphi'(6\varphi)}{\varphi'(\varphi)} = -2.$$

□

Problem 21 (Bending of an elastic rod)

The bending of an elastic rod can be described be the boundary value problem

$$\begin{cases} u'' + \lambda \sin(u) = 0 & \text{in } (0, 2\pi), \\ u'(0) = u'(2\pi) = 0. \end{cases} \tag{3}$$

Find all bifurcation points for problem (3). Sketch the bifurcation diagram near each bifurcation point $(0, \lambda_j)$ with $\lambda_j > 0$ using the bifurcation formulae.

Solution

We consider the Banach spaces $X := \{u \in C^2([0, 2\pi]) : \; u'(0) = u'(2\pi) = 0\}$ (which is a closed subspace of $C^2([0, 2\pi])$) and $Z := C([0, 2\pi])$. We then define the mapping

$$F : X \times \mathbb{R} \to Z, \quad F(u, \lambda) := u'' + \lambda \sin(u).$$

We note that $F(0, \lambda) = 0$ for every $\lambda \in \mathbb{R}$.

As in Problem 18 (a), one shows that F is continuously Fréchet differentiable with respect to u; similar arguments give $F \in C^3(X \times \mathbb{R}, Z)$ with, in particular,

$$\begin{aligned} &F_u(u, \lambda)[v_1] = v_1'' + \lambda \cos(u) v_1, & &F_{u\lambda}(u, \lambda)[v_1] = \cos(u) v_1, \\ &F_{uu}(u, \lambda)[v_1, v_2] = -\lambda \sin(u) v_1 v_2, & &F_{uuu}(u, \lambda)[v_1, v_2, v_3] = -\lambda \cos(u) v_1 v_2 v_3 \end{aligned}$$

for $u, v_1, v_2, v_3 \in X$ and $\lambda \in \mathbb{R}$.

We will first characterize bifurcation points using the Implicit Function Theorem and the Crandall-Rabinowitz Theorem. Then we will use the bifurcation formulae to extract further information.

First Step.

We prove that $\ker F_u(0, \lambda) = \{0\}$ *if and only if* $\lambda \notin \{\frac{1}{4} n^2 : \; n \in \mathbb{N}_0\}$. *Moreover, for* $n \in \mathbb{N}_0$, *we let* $\lambda_n := \frac{1}{4} n^2$ *and show*

$$(\diamondsuit) \qquad \ker F_u(0, \lambda_n) = \operatorname{span} \left\{ \cos\left(\frac{n}{2} \cdot\right) \right\}.$$

For $\lambda \in \mathbb{R}$, we have the equivalence:

$$w \in X, \quad F_u(0, \lambda)[w] = 0 \qquad \Leftrightarrow \qquad w \in C^2([0, 2\pi]), \quad \begin{cases} w'' + \lambda w = 0 & \text{in } (0, 2\pi), \\ w'(0) = w'(2\pi) = 0. \end{cases}$$

A direct calculation shows that this boundary value problem admits nontrivial solutions if and only if $\lambda = \lambda_n$ for some $n \in \mathbb{N}_0$, and that in that case the solution is $w(x) = \alpha \cos\left(\frac{n}{2} x\right)$, $0 \le x \le 2\pi$, with a free parameter $\alpha \in \mathbb{R}$. This is what we intended to prove.

Second Step.

For $n \in \mathbb{N}_0$, we show

$$(\heartsuit) \qquad \operatorname{ran} F_u(0,\lambda_n) = \left\{ z \in Z : \int_0^{2\pi} z(x) \cos\left(\frac{n}{2}x\right) \, \mathrm{d}x = 0 \right\}.$$

For $n = 0$, follow the ideas of the solution of Problem 18 (b). For $n \in \mathbb{N}$, we proceed as follows:

First, we assume $z \in \operatorname{ran} F_u(0,\lambda_n)$, thus there exists $w \in X$ with $F_u(0,\lambda_n)[w] = z$. Hence,

$$w \in C^2([0,2\pi]), \qquad \begin{cases} w'' + \lambda_n w = z & \text{in } (0,2\pi), \\ w'(0) = w'(2\pi) = 0. \end{cases}$$

We now calculate, using integration by parts,

$$\begin{aligned} \int_0^{2\pi} z(x) \cos\left(\frac{n}{2}x\right) \, \mathrm{d}x &= \int_0^{2\pi} w''(x) \cos\left(\frac{n}{2}x\right) \, \mathrm{d}x + \int_0^{2\pi} \lambda_n w(x) \cos\left(\frac{n}{2}x\right) \, \mathrm{d}x \\ &= w'(x) \cos\left(\frac{n}{2}x\right) \Big|_0^{2\pi} + \frac{n}{2} \int_0^{2\pi} w'(x) \sin\left(\frac{n}{2}x\right) \, \mathrm{d}x \\ &\quad + \lambda_n \frac{2}{n} w(x) \sin\left(\frac{n}{2}x\right) \Big|_0^{2\pi} - \lambda_n \frac{2}{n} \int_0^{2\pi} w'(x) \sin\left(\frac{n}{2}x\right) \, \mathrm{d}x = 0. \end{aligned}$$

On the other hand, we consider $z \in Z$ with $\int_0^{2\pi} z(x) \cos\left(\frac{n}{2}x\right) \, \mathrm{d}x = 0$. For arbitrary $\alpha \in \mathbb{R}$, the Picard-Lindelöf Theorem guarantees the existence of a unique $w_\alpha \in C^2([0,2\pi])$ satisfying the initial value problem

$$\begin{cases} w_\alpha'' + \lambda_n w_\alpha = z & \text{in } (0,2\pi), \\ w_\alpha'(0) = 0, w_\alpha(0) = \alpha. \end{cases}$$

We show that the condition on z in $(\heartsuit)$ then yields $w_\alpha'(2\pi) = 0$. To this end, we consider

$$\begin{aligned} w_\alpha'(2\pi) &= (-1)^n \left[w_\alpha'(2\pi) \cos\left(2\pi \frac{n}{2}\right) - w_\alpha'(0) \cos\left(0 \, \frac{n}{2}\right) \right] \\ &= (-1)^n \int_0^{2\pi} \frac{\mathrm{d}}{\mathrm{d}x} \left[w_\alpha'(x) \cos\left(\frac{n}{2}x\right) \right] \, \mathrm{d}x \\ &= (-1)^n \int_0^{2\pi} \left[w_\alpha''(x) \cos\left(\frac{n}{2}x\right) - \frac{n}{2} w_\alpha'(x) \sin\left(\frac{n}{2}x\right) \right] \, \mathrm{d}x \\ &= (-1)^n \int_0^{2\pi} \left[w_\alpha''(x) \cos\left(\frac{n}{2}x\right) + \frac{n^2}{4} w_\alpha(x) \cos\left(\frac{n}{2}x\right) \right] \, \mathrm{d}x \\ &= (-1)^n \int_0^{2\pi} z(x) \cos\left(\frac{n}{2}x\right) \, \mathrm{d}x = 0, \end{aligned}$$

where we used integration by parts and the differential equation solved by w_α in the end. This proves the assertion.

Third Step.

For $n \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$, we show

$$\begin{aligned} C^k([0,2\pi]) &= \operatorname{span}\left\{\cos\left(\frac{n}{2}\,\cdot\right)\right\} \oplus \left\{z \in C^k([0,2\pi]) :\ \int_0^{2\pi} z(x)\cos\left(\frac{n}{2}x\right)\,\mathrm{d}x = 0\right\} \\ &=: N \oplus R_k \end{aligned}$$

and moreover, the projection Q_k of $C^k([0,2\pi])$ along R_k onto N is given by

$$(Q_k z)(x) = \cos\left(\frac{n}{2}x\right)\cdot\frac{1}{\pi}\int_0^{2\pi} z(y)\cos\left(\frac{n}{2}y\right)\,\mathrm{d}y.$$

Q_k is linear. Further, we let $z \in C^k([0,2\pi])$ and calculate for $x \in [0,2\pi]$

$$\begin{aligned} (Q_k^2 z)(x) &= \cos\left(\frac{n}{2}x\right)\cdot\frac{1}{\pi}\int_0^{2\pi}\left[\cos\left(\frac{n}{2}y\right)\cdot\frac{1}{\pi}\int_0^{2\pi} z(t)\cos\left(\frac{n}{2}t\right)\,\mathrm{d}t\right]\cos\left(\frac{n}{2}y\right)\,\mathrm{d}y \\ &= \cos\left(\frac{n}{2}x\right)\cdot\left[\frac{1}{\pi}\int_0^{2\pi} z(t)\cos\left(\frac{n}{2}t\right)\,\mathrm{d}t\right]\cdot\underbrace{\left[\frac{1}{\pi}\int_0^{2\pi}\cos^2\left(\frac{n}{2}y\right)\,\mathrm{d}y\right]}_{=1} = (Q_k z)(x) \end{aligned}$$

and hence, $Q_k^2 = Q_k$. So we have indeed a projection. Then by definition of the sets R_k and N, we infer $R_k = \ker Q_k$ and $N = \operatorname{ran} Q_k$, and the asserted decomposition holds.

Fourth Step.

We prove that $(0,\lambda)$ is a bifurcation point for problem (3) if and only if $\lambda = \lambda_n$ for some $n \in \mathbb{N}_0$. In that case, we obtain a C^1 branch $(\hat{u}_n(s), \hat{\lambda}_n(s))$ of (all) nontrivial solutions in a neighborhood of $(0,\lambda_n)$.

If $\lambda \notin \{\lambda_n : n \in \mathbb{N}_0\}$, then $\ker F_u(0,\lambda) = \{0\}$ by the ($\diamondsuit$), and (as in the solution of Problem 18), the Fredholm alternative and the Continuous Inverse Theorem imply that $F_u(0,\lambda)$ is a linear homeomorphism. Hence, by Corollary III.9 of the Implicit Function Theorem, $(0,\lambda)$ is not a bifurcation point of (3).

If, however, $\lambda = \lambda_n$ for some $n \in \mathbb{N}_0$, the first and third step imply that $\dim\ker F_u(0,\lambda_n) = \operatorname{codim}\operatorname{ran} F_u(0,\lambda_n) = 1$, hence simplicity (S) holds. Transversality (T) also holds since

$$F_{u\lambda}(0,\lambda_n)\left[\cos\left(\frac{n}{2}\,\cdot\right)\right] = \cos\left(\frac{n}{2}\,\cdot\right) \notin \operatorname{ran} F_u(0,\lambda_n),$$

which is a consequence of ($\heartsuit$). The Crandall-Rabinowitz Theorem IV.2 then yields the asserted bifurcation properties.

Fifth Step.

Finally, we apply the bifurcation formulae. For $n \in \mathbb{N}_0$, we let $\varphi_n := \cos\left(\frac{n}{2}\,\cdot\right)$. By the first and second step, φ_n spans both the kernel $\ker F_u(0,\lambda_n)$ and a complement of $\operatorname{ran} F_u(0,\lambda_n)$. We further consider the linear functional

$$\varphi_n' \in Z', \qquad \varphi_n'[z] := \frac{1}{\pi}\int_0^{2\pi} z(x)\cos\left(\frac{n}{2}x\right)\,\mathrm{d}x.$$

Then by the third step, ran $F_u(0,\lambda_n) = \ker \varphi_n'$ and $\varphi_n'[\varphi_n] = 1$.

Observing that $F_{uu}(0,\lambda_n) = 0$, we have $\hat{\lambda}_0'(0) = \hat{\lambda}_0''(0) = 0$ and for $n \in \mathbb{N}$

$$\begin{aligned}
\hat{\lambda}_n'(0) &= 0, \\
\hat{\lambda}_n''(0) &= -\frac{1}{3}\frac{\varphi_n'\left[F_{uuu}(0,\lambda_n)[\varphi_n,\varphi_n,\varphi_n]\right]}{\varphi_n'\left[F_{u\lambda}(0,\lambda_n)[\varphi_n]\right]} = -\frac{1}{3}\frac{-\lambda_n\varphi_n'\left[\varphi_n^3\right]}{\varphi_n'\left[\varphi_n\right]} = \frac{n^2}{12}\frac{\int_0^{2\pi}\varphi_n^4(x)\,\mathrm{d}x}{\int_0^{2\pi}\varphi_n^2(x)\,\mathrm{d}x} \\
&= \frac{3n^2}{16} > 0
\end{aligned}$$

using that $\int_0^{2\pi}\cos^2\left(\frac{n}{2}x\right)\,\mathrm{d}x = \pi$ and $\int_0^{2\pi}\cos^4\left(\frac{n}{2}x\right)\,\mathrm{d}x = \frac{3}{4}\pi$. □