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Bifurcation Theory

Solutions to Problem Sheet 8

Problem 22 (Eigenpairs of the Laplacian on a rectangular domain)

Let $a, b > 0$. Compute all eigenvalues and eigenfunctions of the Laplacian with homogeneous Dirichlet boundary conditions on the rectangular domain $\Omega = (0,a) \times (0,b) \subset \mathbb{R}^2$, i.e. find $\lambda \in \mathbb{C}$, $u \in H_0^1(\Omega)$ such that

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Proceed as follows:

(a) Consider the one dimensional boundary value problem

$$\begin{cases} -u'' = \lambda u & \text{in } (0,a), \\ u(0) = u(a) = 0 \end{cases}$$

and compute all eigenvalues and all eigenfunctions.

(b) Compute all eigenpairs (λ, u) of (1) where u is of the form $u(x,y) = v(x)w(y)$.

(c) Show that there are no other eigenpairs.

Hint: You may use without proof that $\left\{ \sqrt{\frac{2}{a}} \sin\left(\frac{k\pi}{a} x\right) \right\}_{k\in\mathbb{N}}$ *is an orthonormal basis of* $L^2((0,a))$.

Solution

(a) *Assertion: All eigenvalues of this problem are given by* $\lambda_k = \frac{k^2\pi^2}{a^2}$, $k \in \mathbb{N}$. *The corresponding eigenfunctions are unique (up to a multiplicative constant) and they are given by*

$$u_k(x) = \sin\left(\frac{k\pi}{a} x\right).$$

A direct calculation shows, that the solutions of

$$-u'' = \lambda u \quad \text{in } (0,a)$$

are given by

$$u(x) = \begin{cases} c_1 \sin(\sqrt{\lambda}x) + c_2 \cos(\sqrt{\lambda}x), & \lambda > 0, \\ c_1 x + c_2, & \lambda = 0, \\ c_1 \mathrm{e}^{\sqrt{|\lambda|}x} + c_2 \mathrm{e}^{-\sqrt{|\lambda|}x}, & \lambda < 0 \end{cases}$$

where c_1, c_2 are arbitrary constants. We will now distinguish the cases for λ and consider the homogeneous Dirichlet boundary conditions.

Case 1: $\lambda > 0$:
From $u(0) = 0$ we get

$$c_1 \sin(0) + c_2 \cos(0) = 0 \quad \Rightarrow \quad c_2 = 0$$

and $u(a) = 0$ yields with $c_1 \neq 0$

$$c_1 \sin(\sqrt{\lambda}a) = 0 \quad \Rightarrow \quad \sqrt{\lambda}a = k\pi \quad \Rightarrow \quad \lambda = \frac{k^2\pi^2}{a^2}$$

for $k \in \mathbb{N}$.

Case 2: $\lambda = 0$:
From $u(0) = 0$ we get

$$c_1 \cdot 0 + c_2 = 0 \quad \Rightarrow \quad c_2 = 0$$

and $u(a) = 0$ yields with $c_1 \neq 0$

$$c_1 \cdot a = 0.$$

Hence, this case does not occur.

Case 3: $\lambda < 0$:
From $u(0) = 0$ we get

$$c_1 \mathrm{e}^0 + c_2 \mathrm{e}^0 = 0 \quad \Rightarrow \quad c_2 = -c1$$

and $u(a) = 0$ yields with $c_1 \neq 0$

$$c_1 \left(\mathrm{e}^{\sqrt{|\lambda|}a} - \mathrm{e}^{-\sqrt{|\lambda|}a} \right) = 0 \quad \Rightarrow \quad \sqrt{|\lambda|}a = 0.$$

Therefore this case does not occur either.

Summing up, we have the eigenvalues $\lambda_k = \frac{k^2\pi^2}{a^2}$, $k \in \mathbb{N}$ with corresponding eigenfunctions $u_k(x) = \sin\left(\frac{k\pi}{a}x\right)$ and as the normalized eigenfunctions form an orthonormal basis of $L^2((0,a))$, we found all of them.

(b) *Assertion: All eigenpairs (λ, u) of (1) where u is of the form $u(x,y) = v(x)w(y)$ are given by $\lambda_{k,l} = \left(\frac{k^2}{a^2} + \frac{l^2}{b^2}\right)\pi^2$ with corresponding eigenfunctions*

$$u_{k,l}(x,y) = \sin\left(\frac{k\pi}{a}x\right)\sin\left(\frac{l\pi}{b}y\right)$$

with $k, l \in \mathbb{N}$.

With the ansatz $u(x,y) = v(x)w(y)$ we get

$$\Delta u(x,y) = v''(x)w(y) + u(x)v''(y).$$

Thus, the eigenvalue problem reads

$$\begin{cases} -\left(v''(x)w(y) + u(x)v''(y)\right) = \lambda v(x)w(y) \quad \text{in } \Omega, \\ v(0) = v(a) = 0, \\ w(0) = w(b) = 0 \end{cases}$$

or

$$\begin{cases} -\frac{v''(x)}{v(x)} - \frac{w''(y)}{w(y)} = \lambda \quad \text{in } \Omega, \\ v(0) = v(a) = 0, \\ w(0) = w(b) = 0. \end{cases}$$

Consequently, we get the two one-dimensional eigenvalue problems

$$\begin{cases} -v''(x) = \mu v(x) \quad \text{in } (0,a), \\ v(0) = v(a) = 0 \end{cases}$$

and

$$\begin{cases} -w''(y) = \nu w(y) \quad \text{in } (0,b), \\ w(0) = w(b) = 0 \end{cases}$$

with $\lambda = \mu + \nu$. From part (a) we know, that the eigenvalues are given by $\mu_k = \frac{k^2\pi^2}{a^2}$ and $\nu_l = \frac{l^2\pi^2}{b^2}$ for $k, l \in \mathbb{N}$ and the corresponding eigenfunctions are

$$v_k(x) = \sin(\sqrt{\mu_k}x)$$

and

$$w_l(y) = \sin(\sqrt{\nu_l}y),$$

respectively.

Consequently we get as eigenvalues of (1) $\lambda_{k,l} = \left(\frac{k^2}{a^2} + \frac{l^2}{b^2}\right)\pi^2$ with corresponding eigenfunctions

$$u_{k,l}(x,y) = \sin\left(\frac{k\pi}{a}x\right)\sin\left(\frac{l\pi}{b}y\right)$$

with $k, l \in \mathbb{N}$.

(c) *Assertion:* $\left\{\frac{2}{\sqrt{ab}}\sin\left(\frac{k\pi}{a}x\right)\sin\left(\frac{l\pi}{b}y\right)\right\}$ *is an orthonormal basis of* $L^2((0,a)\times(0,b))$ *and all eigenvalues of (1) are given by* $\lambda_{k,l} = \left(\frac{k^2}{a^2}+\frac{l^2}{b^2}\right)\pi^2$.

Let $u \in H_0^1(\Omega)$ be an eigenfunction to the eigenvalue λ, i.e.

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We distinguish two cases.

Case 1: $\lambda \neq \lambda_{k,l}$ for all $k,l \in \mathbb{N}$. Testing the equation by $u_{k,l}$ and integrating twice by parts yields

$$\begin{aligned} 0 &= \int_\Omega -\Delta u(x,y)\cdot u_{k,l}(x,y) - \lambda u(x,y)u_{k,l}(x,y)\,d(x,y) \\ &= \int_\Omega -u(x,y)\Delta u_{k,l}(x,y) - \lambda u(x,y)u_{k,l}(x,y)\,d(x,y) \\ &= (\lambda_{k,l}-\lambda)\int_\Omega u_{k,l}(x,y)u(x,y)\,d(x,y). \end{aligned}$$

Thus, we have for all $k,l \in \mathbb{N}$

$$\int_0^a\int_0^b u(x,y)\sin\left(\frac{k\pi}{a}x\right)\sin\left(\frac{l\pi}{b}y\right)\,dy\,dx = 0.$$

As $\left\{\sqrt{\frac{2}{a}}\sin\left(\frac{k\pi}{a}x\right)\right\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2((0,a))$, this yields for almost all $x \in (0,a)$ and all $l \in \mathbb{N}$

$$\int_0^b u(x,y)\sin\left(\frac{l\pi}{b}y\right)\,dy = 0.$$

Together with the fact that $\left\{\sqrt{\frac{2}{b}}\sin\left(\frac{l\pi}{b}y\right)\right\}_{l\in\mathbb{N}}$ is an orthonormal basis of $L^2((0,b))$ we get

$$u(x,y) = 0$$

for almost all $(x,y) \in (0,a)\times(0,b)$. Hence, there are no other eigenvalues.

Case 2: $\lambda = \lambda_{k,l}$ for some $k,l \in \mathbb{N}$. Define

$$\Psi(x,y) := u(x,y) - \sum_{(k,l)\in I} c_{k,l}u_{k,l}(x,y)$$

with $c_{k,l} = \langle u, u_{k,l}\rangle_{L^2(\Omega)}$ and $I = \{(k,l)\in\mathbb{N}\times\mathbb{N} : \lambda = \lambda_{k,l}\}$. (Note that I is finite due to $\lambda_{k,l}\to\infty$ for $|(k,l)|\to\infty$. [1]). We conclude

$$\begin{cases} -\Delta\Psi = \lambda\Psi & \text{in } \Omega \\ \Psi = 0 & \text{on } \partial\Omega \end{cases}$$

[1] Compare Exercise Class 5

and testing this with $u_{m,n}$ and integrating twice by parts yields

$$\begin{aligned}
0 &= \int_\Omega -\Delta\Psi(x,y)u_{m,n}(x,y) - \lambda\Psi(x,y)u_{m,n}(x,y)\,d(x,y)\\
&= \int_\Omega -\Psi(x,y)\Delta u_{m,n}(x,y) - \lambda\Psi(x,y)u_{m,n}(x,y)\,d(x,y)\\
&= \int_\Omega \Psi(x,y)\lambda_{m,n}u_{m,n}(x,y) - \lambda\Psi(x,y)u_{m,n}(x,y)\,d(x,y)\\
&= (\lambda_{m,n}-\lambda)\int_\Omega \Psi(x,y)u_{m,n}(x,y)\,d(x,y)
\end{aligned}$$

Hence,

$$\langle \Psi, u_{m,n}\rangle_{L^2(\Omega)} = 0 \quad \text{for } (m,n)\notin I$$

as well as

$$\langle \Psi, u_{m,n}\rangle_{L^2(\Omega)} = 0 \quad \text{for } (m,n)\in I$$

and we conclude as in case 1 that $\Psi \equiv 0$, which means that u is a linear combination of eigenfunctions $u_{k,l}$.

□

Problem 23 (Example IV.6 revisited)

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $h \in L^\infty(\Omega)$ and consider the boundary value problem

$$\begin{cases} -\Delta u + \lambda u = h(x)\,|u|^{p-1}\,u & \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{2}$$

Show that for all $p > 2$ there exists a curve of nontrivial solutions $(\widehat{u}(s), \widehat{\lambda}(s))$ of (2) bifurcating from any given simple eigenvalue of $(-\Delta)^{-1} : H_0^1(\Omega) \to H_0^1(\Omega)$.

Hint: Consider

$$-\Delta u + \lambda u = h(x)\chi\left(|u|^{p-1}\,u\right)$$

where $\chi \in C^\infty(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$ *such that* $\chi(z) = z$ *for* $|z| \leq 1$.

Solution

For $n = 1, 2$ this has already been shown in the lecture. Thus we will only prove the assertion for $n \geq 3$.

In order to apply the Crandall-Rabinowitz Theorem, we define

$$F : H_0^1(\Omega) \times \mathbb{R} \to H_0^1(\Omega), \quad F(u,\lambda) := u + \lambda(-\Delta)^{-1}u - (-\Delta)^{-1}\left(h(x)\chi\left(|u|^{p-1}\,u\right)\right).$$

This function is well-defined for $p > 2$ as for $u \in H_0^1(\Omega)$ we have that $u \in L^q(\Omega)$ for all $q \in \left[1, \frac{2n}{n-2}\right)$ and thus $h(x)\chi\left(|u|^{p-1}\,u\right) \in L^q(\Omega)$ for all $q \geq 1$. Consequently[2] we have $(-\Delta)^{-1}\left(h(x)\chi\left(|u|^{p-1}\,u\right)\right) \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ for all $1 \leq q < \infty$. This yields in particular $(-\Delta)^{-1}\left(h(x)\chi\left(|u|^{p-1}\,u\right)\right) \in H_0^1(\Omega)$.

Now solving the modified equation is equivalent to finding zeros of the function F. We have $F(0,\lambda) = 0$ for all $\lambda \in \mathbb{R}$, and F is twice continuously differentiable for $p > 2$ and we have (without proof)

$$F_u(u,\lambda)[\phi] = \phi + \lambda(-\Delta)^{-1}\phi - p(-\Delta)^{-1}\left(h(x)\chi'\left(|u|^{p-1}\,u\right)|u|^{p-1}\,\phi\right)$$

and

$$F_{u\lambda}(u,\lambda)[\phi] = (-\Delta)^{-1}\phi$$

for $\phi \in H_0^1(\Omega)$ and $\lambda \in \mathbb{R}$.

Then we have

$$F_u(0,\lambda_0)[\phi] = \phi + \lambda_0(-\Delta)^{-1}\phi = (id + \lambda_0(-\Delta)^{-1})\phi.$$

[2]For a reference, see Gilbarg Trudigner Lemma 9.17

Hence $F_u(0,\lambda_0)$ is an index-0-Fredholm operator with $\phi \in \ker(F_u(0,\lambda)$ iff

$$(-\Delta)^{-1}\phi = \frac{1}{\lambda_0}\phi$$

for $\lambda_0 \neq 0$ as $(-\Delta)^{-1} : H_0^1(\Omega) \to H_0^1(\Omega)$ is compact, which shows the simplicity condition (S).

If $\ker(F_u(0,\lambda_0)) = \operatorname{span}\{\phi\}$, then we have $F_u(0,\lambda_0) = F_u(0,\lambda_0)^\star$ and consequently

$$\langle F_{u\lambda}(0,\lambda_0)\phi,\phi\rangle = \left\langle (-\Delta)^{-1}\phi,\phi\right\rangle = \int_\Omega \phi^2 > 0$$

and the transversality condition (T) holds.

Hence, the Crandall-Rabinowitz Theorem IV.6 applies and it provides a curve of nontrivial solutions $(\widehat{u}(s),\widehat{\lambda}(s))$ of the modified equation bifurcating from any given simple eigenvalue of $(-\Delta)^{-1} : H_0^1(\Omega) \to H_0^1(\Omega)$. Hence, we know, that $\|\widehat{u}(s)\|_{H_0^1(\Omega)} \to 0$ as $s \to 0$. From the equation we know, that

$$\widehat{u}(s) = -\lambda(-\Delta)^{-1}\widehat{u}(s) + (-\Delta)^{-1}h(x)\chi\left(|u|^{p-1}u\right) \tag{3}$$

and as we have seen before, $(-\Delta)^{-1}\left(h(x)\chi\left(|\widehat{u}(s)|^{p-1}\widehat{u}(s)\right)\right) \to 0$ in $W^{2,q}(\Omega)$ for all $q \geq 1$. Additionally we get from Sobolev's embedding and Gilbarg Trudinger Lemma 9.17 that $(-\Delta)^{-1}\widehat{u}(s) \to 0$ in $W^{2,q}(\Omega)$ for $q \in \left[1,\frac{2n}{n-2}\right)$. Hence $\widehat{u}(s) \to 0$ in $W^{2,q}(\Omega)$ for $q \in \left[1,\frac{2n}{n-2}\right)$.

Now we repeat this argument: If $\widehat{u}(s) \to 0$ in $W^{2,q}(\Omega)$ for $q \in \left[1,\frac{2n}{n-2}\right)$, then we get $(-\Delta)^{-1}\widehat{u}(s) \to 0$ in $W^{2,q}(\Omega)$ for $q \in \left[1,\frac{2n}{(n-4)_+}\right)$ and hence by (3) $\widehat{u}(s) \to 0$ in $W^{2,q}(\Omega)$ for $q \in \left[1,\frac{2n}{(n-4)_+}\right)$. Repeating this procedure, yields finally $\widehat{u}(s) \to 0$ in $L^\infty(\Omega)$.

Thus $\chi\left(|\widehat{u}(s)|^{p-1}\widehat{u}(s)\right) = |\widehat{u}(s)|^{p-1}\widehat{u}(s)$ for s sufficiently small. Namely, we have a curve of nontrivial solutions $(\widehat{u}(s),\widehat{\lambda}(s))$ of the original equation (2) bifurcating from any given simple eigenvalue of $(-\Delta)^{-1} : H_0^1(\Omega) \to H_0^1(\Omega)$.

□