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Bifurcation Theory

Problem Sheet 9

Problem 24 (Necessary and sufficient conditions)

(a) Let $A \in \mathbb{R}^{n\times n}$ be symmetric and $J \in \mathbb{R}^{n\times n}$. Consider

$$F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \quad (x,\lambda) \mapsto Ax - \lambda x + |x|^2 Jx.$$

A necessary condition for bifurcation for $F(x,\lambda) = 0$ at $(0,\lambda_0)$ is that λ_0 is an eigenvalue of A. Show that this condition need not be sufficient.

(b) Let $F(x,\lambda) = x(g(\lambda) + x^2)$ where $g \in C^2(\mathbb{R})$ and $g(\lambda_0) = 0$ and consider the problem

$$(\star) \qquad F(x,\lambda) = 0.$$

(i) Under which conditions on g is $(0,\lambda_0)$ a bifurcation point for $(\star)$?

(ii) Under which conditions is the transversality condition (T) satisfied for $(\star)$?

(iii) Deduce that (T) is sufficient but in general not necessary for bifurcation.

Problem 25 (Bifurcation from ∞ in finite dimensions)

Consider the nonlinear system

$$\begin{cases} (1-\lambda)x_1 + \frac{x_2}{x_1^2+x_2^2} = 0, \\ (1-2\lambda)x_2 + \frac{x_1}{x_1^2+x_2^2} = 0. \end{cases}$$

Show that bifurcation from ∞ occurs for $\lambda_0 = 1$ and $\lambda_0 = \frac{1}{2}$.

Problem 26 (Example IV.11 revisited)

Consider the same example as in the lecture but this time with homogeneous Dirichlet boundary conditions on $\Omega = (0,1)$

$$\begin{cases} -u'' = \lambda u + \frac{a(x,\lambda)u}{1+u^2} \quad \text{in } \Omega, \\ u(0) = u(1) = 0 \end{cases}$$

for $a \in C^1(\overline{\Omega} \times \mathbb{R})$. Assume that $a(x,\lambda) = 0$ whenever x is near 0 or near 1. Use Theorem IV.10 to show that bifurcation from infinity occurs at $\lambda_0 = \pi^2$.