Karlsruhe Institute of Technology (KIT)
Institute for Analysis
Dr. Rainer Mandel
M.Sc. Janina Gärtner, M.Sc. Dominic Scheider

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Bifurcation Theory

Problem Sheet 10

We recall the **Crandall-Rabinowitz Theorem:** Let X, Z be Banach spaces and assume for some $\lambda_0 \in \mathbb{R}$

(i) $F \in C^2(X \times \mathbb{R}, Z)$ and (ii) $F(0,\lambda) = 0$ for all $\lambda \in \mathbb{R}$,

(S) $F_x(0,\lambda_0) \in \mathcal{L}(X,Z)$ is a 1-1-Fredholm operator,

(T) $F_{x\lambda}(0,\lambda_0)[\phi] \notin \operatorname{ran}(F_x(0,\lambda_0))$ where $\ker(F_x(0,\lambda_0)) = \operatorname{span}\{\phi\}$.

Then there is a curve $(\hat{x},\hat{\lambda}) \in C^1((-\varepsilon,\varepsilon), X \times \mathbb{R})$ with $\hat{\lambda}(0) = \lambda_0$, $\hat{x}(0) = 0$, $\hat{x}'(0) = \phi$ and

$$F(\hat{x}(s),\hat{\lambda}(s)) = 0 \qquad \text{for all } s \in (-\varepsilon,\varepsilon).$$

Problem 27 (Crandall-Rabinowitz Theorem via Lyapunov-Schmidt reduction)

(a) Prove the Crandall-Rabinowitz Theorem using Lyapunov-Schmidt reduction. Proceed as follows:

(i) Decompose the spaces X and Z according to the Lyapunov-Schmidt reduction and write down the reduction equation $g(s,\lambda) = 0$, i.e. find a function $g : U \to \mathbb{R}$ where U is a neighborhood of $(0,\lambda_0)$ in $\mathbb{R} \times \mathbb{R}$ such that $F(x,\lambda) = 0$ admits nontrivial solutions (x,λ) bifurcating from $(0,\lambda_0)$ iff $g(s,\lambda) = 0$ admits nontrivial solutions (s,λ) bifurcating from $(0,\lambda_0)$.

(ii) Show that the transversality condition (T) is equivalent to the condition

$$g_{s\lambda}(0,\lambda_0) \neq 0.$$

(iii) Apply the Implicit Function Theorem to conclude the statement.

(b) Now show a slightly different version of the Crandall-Rabinowitz Theorem. Assume that $F \in C^4(X \times \mathbb{R}, Z)$ and that (ii) and (S) hold as well as

(T') $\quad F_{x\lambda}(0,\lambda_0)[\phi], F_{x\lambda\lambda}(0,\lambda_0)[\phi] \in \operatorname{ran}(F_x(0,\lambda_0)), \quad F_{x\lambda\lambda\lambda}(0,\lambda_0)[\phi] \notin \operatorname{ran}(F_x(0,\lambda_0))$.

Prove that in this case $(0,\lambda_0) \in X \times \mathbb{R}$ is a bifurcation point.

Hint: Prove that $g_{s\lambda}(0,\lambda_0) = g_{s\lambda\lambda}(0,\lambda_0) = 0$ *and* $g_{s\lambda\lambda\lambda}(0,\lambda_0) \neq 0$.