Boundary and Eigenvalue Problems:
1st problem sheet

This exercise sheet is about boundary value problems for ordinary differential equations.

Exercise 1

i) Let $f \in C[0,1]$. Prove that the boundary value problem

$$-u'' = f \quad \text{in } (0,1) \quad u(0) = u(1) = 0$$

has a unique solution given by $u(x) = \int_0^1 G(x,t)f(t)\,dt$ where

$$G(x,t) = \begin{cases} 
(1-x)t, & 0 \leq t \leq x \leq 1, \\
(1-t)x, & 0 \leq x \leq t \leq 1. 
\end{cases}$$

ii) Let $f \in C[0,\frac{\pi}{2}]$. Prove that the boundary value problem

$$-u'' - u = f \quad \text{in } (0,\frac{\pi}{2}) \quad u(0) = u\left(\frac{\pi}{2}\right) = 0$$

has a unique solution given by $u(x) = \int_0^{\frac{\pi}{2}} G(x,t)f(t)\,dt$ where

$$G(x,t) = \begin{cases} 
\cos(x)\sin(t), & 0 \leq t \leq x \leq \frac{\pi}{2}, \\
\sin(x)\cos(t), & 0 \leq x \leq t \leq \frac{\pi}{2}. 
\end{cases}$$

Exercise 2

Determine all solutions of the eigenvalue problem

$$-u'' = \lambda u \quad u(0) = u'(0), u(1) = u'(1),$$

where $\lambda \in \mathbb{R}$ is given.
Exercise 3

Let $p \in C^1[0, 1]$ be a positive function, $r \in C[0, 1]$, $\eta_0, \eta_1 \in \mathbb{R}$. Show that the boundary value problem

$$-\frac{d}{dx}(p(x)u'(x)) = r(x) \quad u'(0) = \eta_0, \; u'(1) = \eta_1$$

is solvable if and only if

$$(*) \quad \int_0^1 r(x) \, dx = -p(1)\eta_1 + p(0)\eta_0.$$ 

In case $(*)$ holds find an explicit formula for the solution(s). Is the solution uniquely determined?