Exercise 1

Let $B_1(0)$ be the unit ball in $\mathbb{R}^n$, $f \in C(B_1(0))$, $g \in C(\partial B_1(0))$.

a) Prove that Green’s function for $B_1(0)$ is strictly positive in $B_1(0)$.

b) Show that if $u \in C^2(B_1(0))$ is a solution of the problem

$$-\Delta u = f \quad \text{in } B_1(0)$$
$$u = g \quad \text{on } \partial B_1(0)$$

and $f, g \geq 0$ then $u \geq 0$.

c) Prove that problem (*) has at most one solution in $C^2(B_1(0))$.

Exercise 2

Use Poisson’s integral formula (Corollary 14) to prove that every positive harmonic function $u \in C^2(B_R(0))$ satisfies the inequality

$$\frac{R^{n-2}(R - |x|)}{(R + |x|)^{n-1}} u(0) \leq u(x) \leq \frac{R^{n-2}(R + |x|)}{(R - |x|)^{n-1}} u(0) \quad \forall x \in B_R(0).$$

Show that this inequality does not hold if the positivity assumption is dropped.
Exercise 3

In this exercise we are interested in solving the boundary value problem

\[-\Delta u(x) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty)\]

\[\lim_{x \to x'(0)} u(x) = g(x') \quad \text{on } \mathbb{R}^n \times \{0\}\]

for \(g: \mathbb{R}^n \to \mathbb{R}\) continuous and bounded.

a) Prove that the function

\[u(x', x_{n+1}) = \frac{2}{\omega_{n+1}} \int_{\mathbb{R}^n} \frac{x_{n+1}}{|x' - y|^2 + x_{n+1}^2} g(y') \, dy'\]

defines a solution for \((**)\) where \(\omega_n\) denotes the surface area of \(\partial B_1(0)\). You may use

\[1 = \frac{2}{\omega_{n+1}} \int_{\mathbb{R}^n} \frac{x_{n+1}}{|x' - y|^2 + x_{n+1}^2} \, dy' \quad \text{for all } x_{n+1} > 0, x' \in \mathbb{R}^n\]

b) Determine the Green’s function for the ball \(B_R(z) \subset \mathbb{R}^n\) for arbitrary \(z \in \mathbb{R}^n, R > 0\).

c) Let \(\Gamma_R\) be the Green’s function for \(B_R(Re_n)\) where \(e_n\) denotes the \(n\)-th unit vector in \(\mathbb{R}^n\). Prove

\[\Gamma_R(x, y) \to \gamma(x - y) - \gamma(x - \hat{y}) \quad \text{as } R \to \infty\]

where \(\gamma\) is the fundamental solution of \(-\Delta\) and \(\hat{y} = (y_1, \ldots, y_{n-1}, -y_n)\).

Supplementary exercise

Let \(\gamma: \mathbb{R}^n \to \mathbb{R}\) be the fundamental solution of \(-\Delta\) in \(\mathbb{R}^n\) (see Lemma 4 of the lecture) and let \(\nu_j\) denote the \(j\)-th component of the outer unit normal field \(\nu\) on \(\partial B_1(0)\).

a) Show that the following identity holds for \(x \in B_1(0)\):

\[\oint_{\partial B_1(0)} \gamma(x - y) \nu_j \, ds_y = \int_{B_1(0)} -\frac{\partial}{\partial x_j} \gamma(x - y) \, dy.\]

b) Use a) to prove

\[\oint_{\partial B_1(0)} -\frac{\partial}{\partial x_i} \gamma(x - y) \nu_j \, ds_y = \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \int_{B_1(0)} \gamma(x - y) \, dy.\]

Remark: This fact was used in the proof of Proposition 13.