Let $X$ be a set, e.g. $X = \mathbb{R}^n$, and let $\mathcal{P}(X) = \{\text{all subsets of } X\}$.

**Definition L.1 (σ-algebra)** A system of sets $\mathcal{M} \subset \mathcal{P}(X)$ is called a $\sigma$-algebra over $X$ if

(i) $X \in \mathcal{M}$

(ii) $A \in \mathcal{M} \implies X \setminus A \in \mathcal{M}$

(iii) $A_i \in \mathcal{M} \quad \forall i \in \mathbb{N} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$

**Definition L.2 (positive measure)** Let $\mathcal{M}$ be $\sigma$-algebra over $X$. A map $\mu : \mathcal{M} \to [0, \infty)$ is called a positive measure, if

$$A_i \in \mathcal{M} \quad \forall i \in \mathbb{N} \text{ with } A_i \cap A_j = \emptyset \text{ for } i \neq j \implies \mu \left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu(A_i).$$

Let $I = I^1 \times \cdots \times I^n = (a_1, b_1) \times \cdots \times (a_n, b_n)$ be an open intervall in $\mathbb{R}^n$. Then the volume of $I$ is defined by $|I| = (b_1 - a_1) \cdot \cdots \cdot (b_n - a_n)$. The same applies if one or more of the component-intervals $(a_i, b_i)$ are replaced by closed, semi-closed, semi-open intervals.

**Definition L.3 (outer measure on $\mathbb{R}^n$)** Let $A \subset \mathbb{R}^n$ be an arbitrary set. Then

$$\lambda(A) := \inf \left\{ \sum_{i=1}^{\infty} |I_i| : A \subset \bigcup_{i=1}^{\infty} I_i \text{ and } I_i \text{ bounded interval } \forall i \in \mathbb{N} \right\}$$

is called the outer measure of the set $A$.

**Remark:** $\lambda$ is not a positive measure on $\mathcal{P}(\mathbb{R}^n)$.

**Definition L.4 (Lebesgue σ-algebra, Caratheodory)** A set $A \subset \mathbb{R}^n$ is called Lebesgue-measurable (short: $A \in \mathcal{L}(\mathbb{R}^n)$) if

$$\lambda(E) = \lambda(A \cap E) + \lambda(A^c \cap E) \quad \forall E \subset \mathbb{R}^n.$$  

If $X \subset \mathbb{R}^n$ is Lebesgue-measurable, then let $\mathcal{L}(X) = \{A \subset X : A \text{ is Lebesgue-measurable}\}$.

**Theorem L.5** $\mathcal{L}(\mathbb{R}^n)$ is a $\sigma$-algebra. The outer measure $\lambda$ (see Definition L.3) is invariant under Euclidean motions and if it is restricted to $\mathcal{L}(\mathbb{R}^n)$ then it becomes a positive, complete measure on $\mathcal{L}(\mathbb{R}^n)$. 
In the following let $X \subset \mathbb{R}^n$ be a Lebesgue-measurable set. For $f : X \to \mathbb{R} = [-\infty, \infty]$ let $f^+ = \max\{f, 0\}, f^- = -\min\{f, 0\}$. Hence $f = f^+ - f^-$. 

**Definition L.6 (measurable functions)**

(i) A function $f : X \to \mathbb{R}$ is called measurable, if $f^{-1}((\alpha, \infty]) \in \mathcal{L}(X)$ for all $\alpha \in \mathbb{R}$,

(ii) A function $s : X \to \mathbb{R}$ is called an elementary function, if $s$ possesses only finitely many values $\alpha_1, \ldots, \alpha_k$. In this case

$$s = \sum_{i=1}^{k} \alpha_i \chi_{A_i}, \quad A_i = s^{-1}(\alpha_i).$$

**Definition L.7 (Lebesgue-integral for non-negative functions)**

(i) Let $s = \sum_{i=1}^{k} \alpha_i \chi_{A_i}$ be a measurable elementary function. Then

$$\int_X s \, dx := \sum_{i=1}^{k} \alpha_i \lambda(A_i)$$

is called the Lebesgue-integral of $s$ over $X$.

(ii) Let $f : X \to [0, \infty]$ be measurable. Then

$$\int_X f \, dx := \sup_{s \in \mathcal{S}} \int_X s \, dx, \quad \mathcal{S} = \{s : X \to \mathbb{R} \text{ measurable elementary function }, 0 \leq s \leq f\}$$

is called the Lebesgue-integral of $f$ over the set $X$.

**Definition L.8 (Lebesgue-integral for real- or complex-valued functions)** Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

$$L^1(X) := \{f : X \to \mathbb{K} \text{ measurable } : \int_X |f| \, dx < \infty\}.$$ 

For $f \in L^1(X)$ let $f_1 = \Re f, f_2 = \Im f$. Then

$$\int_X f \, dx := \int_X f_1^+ \, dx - \int_X f_1^- \, dx + i \left( \int_X f_2^+ \, dx - \int_X f_2^- \, dx \right)$$

is called the Lebesgue-integral of $f$ over the set $X$.

**Definition L.9 (f = g a.e.)** Let $f, g : X \to \mathbb{K}$ be measurable. Then we say $f = g$ almost everywhere, if there exists a set $N$ of measure 0 such that $f(x) = g(x) \forall x \in X \setminus N$. Equality almost everywhere is an equivalence relation.
Definition L.10 (The space \( L^p(X) \))

(a) For \( 1 \leq p < \infty \) let

\[
L^p(X) = \{ u : X \to \mathbb{R} \text{ measurable: } \int_X |u|^p \, dx < \infty \}.
\]

(b) For \( p = \infty \) let

\[
L^\infty(X) = \{ u : X \to \mathbb{R} \text{ measurable: } \text{ess sup}_X |u| < \infty \},
\]

where \( \text{ess sup}_X v = \inf \{ s \in \mathbb{R} : v(x) \leq s \text{ for almost all } x \in X \} \).

Definition L.11 (Norm on \( L^p(X) \)) For \( 1 \leq p < \infty \) let

\[
\|u\|_p := \left( \int_X |u|^p \, dx \right)^{1/p}
\]

and

\[
\|u\|_\infty := \text{ess sup}_X |u|.
\]

Then \( (L^p(X), \| \cdot \|_p) \) is a Banach space.

Theorem L.12 (Minkowski and Hölder inequalities)

(i) \( \|u + v\|_p \leq \|u\|_p + \|v\|_p \) for all \( u, v \in L^p(X) \).

(ii) Let \( 1 \leq p, q \leq \infty \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Then

\[
\int_X |uv| \, dx \leq \|u\|_p \|v\|_q
\]

for all \( u \in L^p(X) \) and all \( v \in L^q(X) \).

Theorem L.13 Let \( 1 \leq p \leq \infty \) and \( u \in L^p(X) \). If \( (u_k)_{k \in \mathbb{N}} \) is a sequence of functions in \( L^p(X) \) such that \( \lim_{k \to \infty} \|u_k - u\|_p = 0 \) then there exists a subsequence \( (u_{k_l})_{l \in \mathbb{N}} \) such that

\[
\lim_{l \to \infty} u_{k_l}(x) = u(x) \text{ for almost all } x \in X.
\]

Theorem L.14 (Monotone convergence) Let \( (u_k)_{k \in \mathbb{N}} \) be a sequence of measurable functions on \( X \) such that

\[
0 \leq u_1 \leq u_2 \leq u_3 \leq \ldots.
\]

Then \( u(x) := \lim_{k \to \infty} u_k(x) \) exists for almost all \( x \in X \) and

\[
\lim_{k \to \infty} \int_X u_k \, dx = \int_X u \, dx.
\]
Theorem L.15 (Dominated convergence) Let \((u_k)_{k \in \mathbb{N}}\) be a sequence of measurable functions on \(X\). If there exists \(w \in L^1(X)\) such that \(|u_k(x)| \leq w(x)\) for almost all \(x \in X\) and all \(k \in \mathbb{N}\) and if \(u(x) := \lim_{k \to \infty} u_k(x)\) exists almost everywhere in \(X\) then
\[
\lim_{k \to \infty} \int_X u_k \, dx = \int_X u \, dx.
\]

Theorem L.16 (Fatou’s Lemma) Let \((u_k)_{k \in \mathbb{N}}\) be a sequence of measurable functions on \(X\) such that \(u_k(x) \geq 0\) almost everywhere on \(X\). Then
\[
\int_X \lim \inf_{k \in \mathbb{N}} u_k \, dx \leq \lim \inf_{k \in \mathbb{N}} \int_X u_k \, dx.
\]

Theorem L.17 Let \(1 \leq p < \infty\). Then the set of continuous functions with compact support \(C_c(X)\) is dense in \(L^p(X)\).

Theorem L.18 (Dual space of \(L^p(X)\)) Let \(1 \leq p < \infty\) and let \(\phi : L^p(X) \to \mathbb{R}\) be a continuous linear functional. Then there exists a unique \(v \in L^q(X), \frac{1}{p} + \frac{1}{q} = 1\), such that
\[
\phi(u) = \int_X uv \, dx \text{ for all } u \in L^p(X).
\]
For short: \((L^p(X))^* = L^q(X)\).

Note: In general the theorem fails for \(p = \infty\), i.e., \((L^\infty(X))^* \nsubseteq L^1(X)\).