

1. Exercise sheet - Computer assisted methods for partial differential equations

Exercise 1

Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and for some $v, w \in \mathbb{R}$, $v < w$ let $F(v)F(w) < 0$. Prove, that there exists $A \in \mathbb{R}$, $A \neq 0$, such that the operator T defined by

$$Tu = u + AF(u) \quad \text{for } u \in \mathbb{R}$$

satisfies $T([v, w]) \subset [v, w]$. From this, deduce the intermediate value theorem for continuous functions.

Exercise 2

a) Let $\Omega = (0, 1)^2$. Calculate an explicit value $C > 0$, such that the inequality

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)} \tag{1}$$

holds true for all $u \in H_0^1(\Omega)$.

b) Let now $\Omega \subset \mathbb{R}^2$ be an arbitrary bounded and open set. Prove the existence of a constant $C > 0$ having property (1) for all $u \in H_0^1(\Omega)$.

Exercise 3

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and for $u \in H^2(\Omega) \cap H_0^1(\Omega)$ let $-\Delta u \geq 0$ a.e. in Ω . Prove that $u \geq 0$ a.e. in Ω .

Hint: It might be useful to consider the function $v^- = -\min\{v, 0\} \in H_0^1(\Omega)$ (for $v \in H_0^1(\Omega)$) with weak gradient

$$\nabla v^- = \begin{cases} 0 & \text{in } \{x \in \Omega : v(x) \geq 0\} \\ -\nabla v & \text{in } \{x \in \Omega : v(x) < 0\}. \end{cases}$$