Karlsruhe Institute for Technology
Institute for Analysis
Prof. Dr. Michael Plum
Dipl.-Math. Dagmar Roth

SS 2010

4. Exercise sheet - Computer assisted methods for partial differential equations

Exercise 9

Compute constants C_0, C_1, C_2 such that the explicit embedding inequality

$$\|u\|_\infty \leq C_0\|u\|_{L^2(\Omega)} + C_1\|\nabla u\|_{L^2(\Omega)} + C_2\|u_{xx}\|_{L^2(\Omega)} \quad (u \in H^2(\Omega))$$

is valid in the cases

a) $\Omega = \{x \in \mathbb{R}^n : \ |x| < \rho\}$, $\rho > 0$, $n \in \{2,3\}$

b) $\Omega = \{x \in \mathbb{R}^n : \ 0 < x_i < l_i, \ i = 1, \ldots, n\}$, $l_i > 0$, $n \in \{2,3\}$

Exercise 10

Let $\Omega \subset \mathbb{R}^3$ be an open set with C^2-boundary and $u \in C^2(\bar{\Omega})$, $u|_{\partial\Omega} = 0$. Prove that on $\partial\Omega$

$$-\Delta u \frac{\partial u}{\partial \nu} + \nu^T u_{xx} \nabla u = 2H \left(\frac{\partial u}{\partial \nu}\right)^2,$$

where ν denotes the outer unit normal field and H the mean curvature.

Hint: For $x_0 \in \partial\Omega$ choose a coordinate system, such that the (x_1, x_2)-plane is tangential to $\partial\Omega$ in x_0 and the positive x_3-axis points into the direction of $\nu(x_0)$. Then in a neighborhood of x_0 the boundary $\partial\Omega$ can be written as graph of a C^2-function f and the mean curvature in x_0 is given by

$$H(x_0) = \frac{1}{2}\operatorname{tr}(f_{xx}(x_0)).$$

Please turn over!

Exercise 11

Let $\Omega \in \mathbb{R}^n$ ($n = 2, 3$) a domain with C^2-boundary. Furthermore let $F_0, F_1 > 0$ and $f : \bar{\Omega} \to \mathbb{R}^n$ be known such that

$$\begin{aligned} f^T(x)\nu(x) \geq (n-1)H(x) &\quad (x \in \partial\Omega) \\ |f(x)| \leq F_0 &\quad (x \in \bar{\Omega}) \\ \lambda_{\max}[-\operatorname{div} f(x) I + Df(x) + Df(x)^T] \leq F_1 &\quad (x \in \Omega) \end{aligned}$$

where $\lambda_{\max}(M)$ denotes the biggest eigenvalue of the symmetric matrix M.

a) For $u \in H^2(\Omega) \cap H_0^1(\Omega)$ prove the inequality

$$\|u_{xx}\|_{L^2}^2 \leq \|\Delta u\|_{L^2}^2 + 2F_0\|\nabla u\|_{L^2}\|\Delta u\|_{L^2} + F_1\|\nabla u\|_{L^2}^2.$$

Hint: $\left(\frac{\partial u}{\partial \nu}\right)^2 f \cdot \nu = [2(f \cdot \nabla u)\nabla u - |\nabla u|^2 f] \cdot \nu$ on $\partial\Omega$.

b) Use a) to conclude $\|u_{xx}\|_{L^2} \leq K_2\|L[u]\|_{L^2}$ where

$$K_2 = \sqrt{\kappa^2 + 2F_0K_1\kappa + F_1K_1^2},$$

where $\kappa = 1 + K_0\|c\|_\infty$ and K_0, K_1 as defined in the lecture.

The exercise sheet will be discussed in the exercise on June, 11th.