Problem 13
Let $I := (-1, 1)$ and $L : H^1(I) \rightarrow \mathbb{R}$ be given by
\[ L[u] = \int_{-1}^{1} u'(x)^2 \, dx, \quad u \in H^1(I). \]
Compute all minimizers of $L|_M$ where
\begin{enumerate}[a)]  
  \item $M = \left\{ u \in H^1_0(I) : \int_{-1}^{1} u^2(x) \, dx = 1 \right\}$,
  \item $M = \left\{ u \in H^1(I) : \int_{-1}^{1} u^2(x) \, dx = 1 \text{ and } \int_{-1}^{1} u(x) \, dx = 0 \right\}$.
\end{enumerate}

Hint: Without proof, you may use that the minimizers belong to $C^2[-1, 1]$.

Problem 14
Let $\Omega$ be a bounded and open subset of $\mathbb{R}^n$, $2 < p < 2^*$ where $2^*$ is defined as in Problem 11 and $H := H^1_0(\Omega) \setminus \{0\}$. Prove that the functional $L : H \rightarrow \mathbb{R}$ given by
\[ L[u] = \frac{\int_{\Omega} |\nabla u(x)|^2 \, dx}{\left( \int_{\Omega} |u(x)|^p \, dx \right)^{\frac{2}{p}}} \]
has a minimizer $u_0 \in H$ and find a PDE for $u_0$.

Hint: Use Problem 11.

To be discussed in the Problem session on Tuesday, January 19, 2010.

We wish you a merry Christmas and a happy new year 2010!