Problem 15
Let $X$ be a Banach space, $U$ an open subset of $X$ and $L, m \in C^1(X, \mathbb{R})$. Furthermore, let $c \in \mathbb{R}, M := \{u \in U : m[u] \leq c\}$ and $u_0 \in M$ be such that

$$L[u_0] = \inf_{u \in M} L[u].$$

Prove that there exist multipliers $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $(\lambda_1, \lambda_2) \neq (0, 0)$ and

$$\lambda_1DL[u_0] = \lambda_2Dm[u_0].$$

Problem 16
Let $\Omega \neq \emptyset$ be a bounded and open subset of $\mathbb{R}^n$ and $2 < p < 2^*$ where $2^*$ is defined as in Problem 11.

a) Prove that the functional $L : H^1_0(\Omega) \longrightarrow \mathbb{R}$ given by

$$L[u] = \int_\Omega |u|^p \, dx, \quad u \in H^1_0(\Omega)$$

has a maximizer $u_0 \in M$ on the set

$$M := \left\{ u \in H^1_0(\Omega) : \int_\Omega (|\nabla u|^2 + u^2) \, dx = 1 \right\}.$$

b) By means of part a), show that the boundary value problem

$$-\Delta u + u = |u|^{p-2}u \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega$$

has a nontrivial weak solution $u \in H^1_0(\Omega) \setminus \{0\}$.

To be discussed in the Problem session on Tuesday, January 19, 2010.