11th Problem Sheet
Variational Methods and Applications to PDEs

Problem 18
Let $\Omega$ be a bounded, open subset of $\mathbb{R}^n$ with $\partial \Omega \in C^1$ and let the functional $L : H^1(\Omega)^m \rightarrow \mathbb{R}$ be given by

$$L[u] = \frac{1}{2} \sum_{j=1}^{m} \int_{\Omega} |\nabla u_j|^2 \, dx$$

for $u = (u_1, \ldots, u_m) \in H^1(\Omega)^m$.

Furthermore, let $g \in L^2(\partial \Omega)^m$ and $M := \{ u \in H^1(\Omega)^m : u|_{\partial \Omega} = g, |u| = 1 \text{ a.e. on } \Omega \}$ (note that here, $|\cdot|$ denotes the Euclidean norm both in $\mathbb{R}^n$ and $\mathbb{R}^m$).

a) Show that if $M \neq \emptyset$, then $L|_M$ has a minimizer $u \in M$.

b) Prove that if $u \in M$ is a minimizer of $L|_M$, then $u$ satisfies

$$\sum_{j=1}^{m} \int_{\Omega} \nabla u_j \cdot \nabla v_j \, dx = \sum_{j=1}^{m} \int_{\Omega} |\nabla u_j|^2 u \cdot v \, dx$$

for any $v \in H^1_0(\Omega)^m \cap L^\infty(\Omega)^m$.

c) Find a strong formulation for the boundary value problem $(\ast)$.

Instructions:

• For part a), use that there exist a compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ and a bounded linear operator $T : H^1(\Omega) \rightarrow L^2(\partial \Omega)$ such that $Tu = u|_{\partial \Omega}$ for any $u \in H^1(\Omega) \cap C(\Omega)$. In this sense, we write $u|_{\partial \Omega} := Tu$ for all $u \in H^1(\Omega)$. $T$ is the so-called trace-operator.

• For part b), let $v \in H^1_0(\Omega)^m \cap L^\infty(\Omega)^m$ and consider $w_\tau = \frac{u + \tau v}{|u + \tau v|}$ and $\ell(\tau) = L[w_\tau]$. Later, use (and prove!) that $u$ satisfies $\sum_{j=1}^{m} u_j \nabla u_j = 0$.

– please turn over! –
Problem 19
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $A \in C(\overline{\Omega}, \mathbb{R}^{n \times n})$, $b \in C(\overline{\Omega}, \mathbb{R}^n)$ and $f \in C(\overline{\Omega} \times \mathbb{R})$. We assume that $A$ is uniformly positive definite, i.e. $\xi^\top A(x)\xi \geq a|\xi|^2$ for any $x \in \Omega$ and any $\xi \in \mathbb{R}^n$ with some $a > 0$, and that there exists a function $\varphi \in C^1(\overline{\Omega})$ such that $\nabla \varphi(x) = A^{-1}(x)b(x)$ for any $x \in \Omega$.

a) Prove that the boundary value problem
\begin{align*}
-\text{div}(A(x)\nabla u) + b(x) \cdot \nabla u + f(x, u) &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega
\end{align*}
can be written as a variational problem by introducing a new unknown $w$ such that $u = e^w w$.

b) Suppose now that $f(x, u) = c(x)u - r(x)$, where $c, r \in C(\overline{\Omega})$ and $c \geq 0$ in $\Omega$. Prove that a minimizer for the variational problem in part a) exists.

To be discussed in the problem session on Tuesday, February 2, 2010.