12th Problem Sheet

Variational Methods and Applications to PDEs

Problem 20
Check whether the following mappings satisfy the Palais-Smale condition:

a) \( f : \mathbb{R} \rightarrow \mathbb{R}, \ f(x) = x \sin(x), \)

b) \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \ f(x, y) = (\cos(x) + y^2, 2xy), \)

c) \( L : L^2[0, 1] \rightarrow \mathbb{R}, \ L[u] = \int_0^{1/2} u^2 \, dx. \)

Problem 21
Prove the following statements:

a) If \( f \in C^1(\mathbb{R}^N, \mathbb{R}) \) and if the mapping
\[ \mathbb{R}^N \ni x \mapsto |f(x)| + |\nabla f(x)| \in \mathbb{R} \]
is coercive, then \( f \) satisfies the Palais-Smale condition.

b) If \( f \in C^1(\mathbb{R}, \mathbb{R}) \) is bounded from below and satisfies the Palais-Smale condition, then \( f \) is coercive and attains its infimum.

To be discussed in the problem session on Tuesday, February 2, 2010.