

# Surface Gap Solitons for the NLS

*Dedicated to the memory of Wolfgang Walter (May 2, 1927 – June 26, 2010)*

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Institute for Analysis

# Ground states for the periodic NLS

Purely periodic problems:  $1 < p < \frac{n+2}{(n-2)_+}$

$$\underbrace{(-\Delta + V_i(x))}_{=:L_i} u = \Gamma_i(x) |u|^{p-1} u \text{ in } \mathbb{R}^n, \quad i = 1, 2$$

$V_1, V_2, \Gamma_1, \Gamma_2$  1-periodic,  $\text{ess inf } \Gamma_i > 0$  (focusing)

Pankov:  $0 \notin \sigma(L_i) \Rightarrow \exists$  ground state  $w_i$

$$c_i = \inf_{u \in \mathcal{M}_i} \int_{\mathbb{R}^n} \frac{|\nabla u|^2 + V_i(x) u^2}{2} - \frac{\Gamma_i(x) |u|^{p+1}}{p+1} dx$$

where  $\mathcal{M}_i =$  Nehari-manifold

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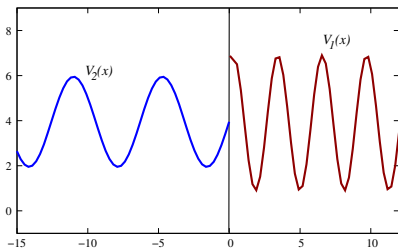
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# Interface problems for the NLS

$$(-\Delta + V(x))u = \Gamma(x)|u|^{p-1}u \text{ in } \mathbb{R}^n$$

$$V(x) = \begin{cases} V_1(x), & x_1 > 0, \\ V_2(x), & x_1 < 0, \end{cases}$$

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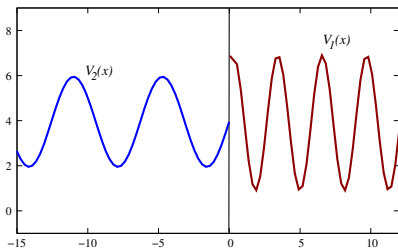
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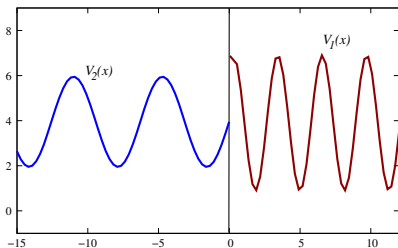
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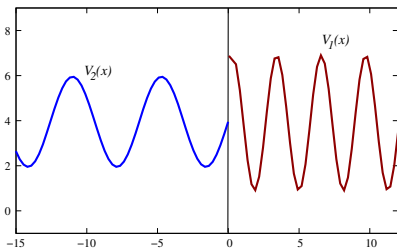
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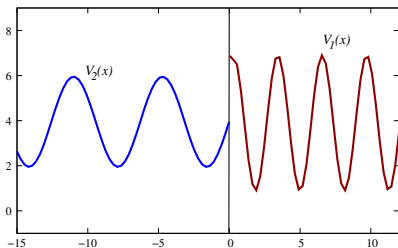
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T. Dohnal and D. Pelinovsky: *Surface gap solitons at a nonlinearity interface*.

SIAM J. Appl. Dyn. Syst. **7** (2008), 249–264.

# Assumptions and observations

## Spectrum of $L$ :

$$\sigma(L) = \sigma(L_1) \cup \sigma(L_2) \cup \{\text{possibly eigenvalues}\}$$

Assume in the following:  $0 < \inf \sigma(L)$

## Energy-levels:

$$c \leq \min\{c_1, c_2\}$$

There are sequences  $\{u_k\}_{k \in \mathbb{N}}$  in  $\mathcal{M}$  with

- $J[u_k] \rightarrow c_1$  (likewise for  $c_2$ )
- $J'[u_k] \rightarrow 0$
- $u_k \rightarrow 0$  as  $k \rightarrow \infty$

## Non-existence:

- if  $V_1 \leq V_2$ ,  $\Gamma_1 \geq \Gamma_2$  (strict somewhere)
- minimizing sequence wants to look like  $w_1(x - te_1)$ ,  $t$  large
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# Abstract result

## Theorem (Dohnal, Plum, R., 2010)

*Assume  $0 < \inf \sigma(L)$  and  $1 < p < \frac{n+2}{(n-2)_+}$ . If  $c < \min\{c_1, c_2\}$  then  $c$  is attained, i.e., there exists a ground state for the interface problem*

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Proof idea: Take minimizing sequence  $u_k \in \mathcal{M}$  s.t.

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Then  $0 < d_1 \leq \|u_k\| \leq d_2$ , i.e.  $u_k \rightharpoonup u_0$  (for a subsequ.). **Show  $u_0 \neq 0$ .**

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 If  $\liminf \|u_k\|_{H^1(S_\delta)} = 0$ ,  $S_\delta = [-\delta, \delta] \times \mathbb{R}^{n-1} \Rightarrow c \geq \min\{c_1, c_2\}$ . **Impossible!**

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Reason:

$$u_k = \underbrace{\chi_{[\delta, \infty) \times \mathbb{R}^{n-1}} u_k}_{=: v_k} + \underbrace{\chi_{(-\infty, -\delta] \times \mathbb{R}^{n-1}} u_k}_{=: w_k} + \underbrace{\text{rest}}_{\rightarrow 0}$$

Hence

$$J[u_k] = J_1[v_k] + J_2[w_k] + o(1) \geq \min\{c_1, c_2\} + o(1)$$

## Consequence: a practical criterion

Find one function  $u \in \mathcal{M}$  such that

$$J[u] < \min\{c_1, c_2\}$$

Candidate: If  $c_1 \leq c_2$  take

$$u_t(x) := w_1(x - te_1), \quad t \rightarrow \infty$$

Scale: Choose  $s > 0$  such that  $su_t \in \mathcal{M}$  ( $\mathcal{M}$  is a top. sphere)

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Compute:

$$\begin{aligned} J[su_t] &= c_1 \frac{c_1 + \frac{1}{2} \int_{\mathbb{R}^n} (V_2 - V_1) u_t^2 dx (1 + o(1))}{c_1 + \frac{1}{p+1} \int_{\mathbb{R}^n} (\Gamma_2 - \Gamma_1) |u_t|^{p+1} dx (1 + o(1))} \\ &< c_1 \text{ for large } t \gg 1 \text{ provided} \end{aligned}$$

$$(p+1) \int_{\mathbb{R}^n} (V_2 - V_1) u_t^2 dx < 2 \int_{\mathbb{R}^n} (\Gamma_2 - \Gamma_1) |u_t|^{p+1} dx$$

How to verify the criterion (assuming  $c_1 \leq c_2$ ):

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- Make sure  $c_1 \leq c_2$
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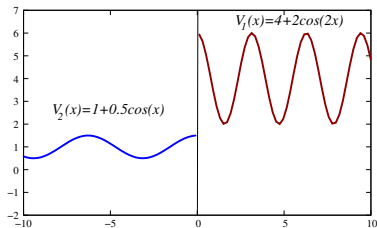
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$$V_1(x) = k^2 V_2(x), \quad k \in \mathbb{N},$$

$$\Gamma_1(x) = \gamma \Gamma_2(kx), \quad \gamma > 0$$



$$\text{ground states: } w_1(x) = \left(\frac{k}{\gamma}\right)^{\frac{2}{p-1}} w_2(kx)$$

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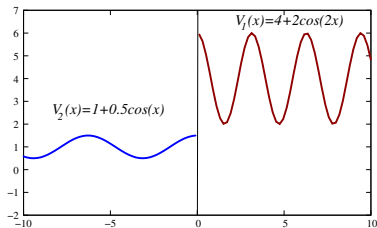
Verify criterion:

- $c_1 \leq c_2$  : take  $\gamma$  so large that  $\gamma^{\frac{4}{p-1}} > k^{\frac{n-2-p(n-2)}{p-1}}$   $\Rightarrow \exists$  ground state
- $V_2 < V_1$  : take  $k$  so large that  $\sup V_2 < k^2 \inf V_2$

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## Example 2 (large contrast in $\Gamma_1/\Gamma_2$ )

Let  $V_2 < V_1$ . How to ensure  $c_1 \leq c_2$ ?

Assume:  $\Gamma_1(x) \geq \beta_0 > 0$ ,  $\beta_0$  large

$$c_1 = c(V_1, \Gamma_1) \leq c(V_1, \beta_0) \quad (\text{monotonicity})$$

$$= \beta_0^{\frac{-2}{p-1}} c(V_1, 1)$$

$$\leq c(V_2, \Gamma_2) = c_2 \quad (\text{provided } \beta_0 \text{ large})$$

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## Example 3 – Dislocation ( $n = 1$ )

$$\left(-\frac{d^2}{dx^2} + V(x) - \lambda\right)u = \Gamma(x)|u|^{p-1}u \text{ on } \mathbb{R}$$

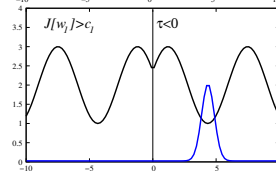
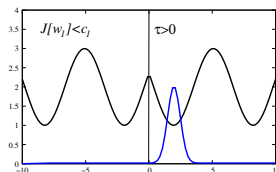
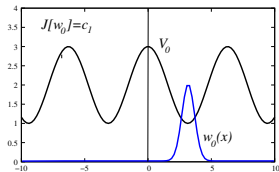
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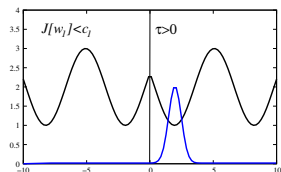
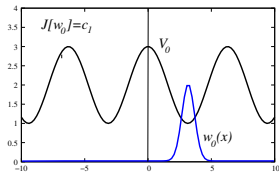


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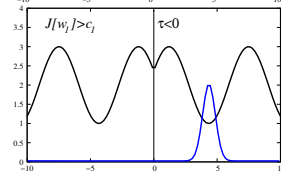
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Heuristically:

$$V'(0) = 0, V''(0) < 0, \tau > 0$$

gives energetic advantage!



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Check criterion for large  $t$ :

$$(\rho + 1) \underbrace{\int_{-\infty}^0 \delta V(x) w_0^2(x + \tau - t) dx}_{=: I_1} < 2 \underbrace{\int_{-\infty}^0 \delta \Gamma(x) |w_0(x + \tau - t)|^{\rho+1} dx}_{=: I_2}$$

Use asymptotics:

$$\frac{w_0(x)}{p_-(x)e^{kx}} \xrightarrow{x \rightarrow -\infty} \text{const.}, \quad p_{\pm} e^{\mp kx} = \text{Bloch mode for } -\frac{d^2}{dx^2} + V(x) - \lambda$$

$$I_1 = \text{const.}^2 (1 + o(1)) e^{2k(\tau-t)} \int_{-1}^0 \delta V(x) p_-(x + \tau)^2 e^{2kx} dx$$

$$I_2 = \text{const.}^{\rho+1} (1 + o(1)) e^{(\rho+1)k(\tau-t)} \int_{-1}^0 \delta \Gamma(x) p_-(x + \tau)^{\rho+1} e^{(\rho+1)kx} dx$$

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$$I_1 = \text{const.}^2 (1 + o(1)) e^{2k(\tau-t)} \int_{-1}^0 \delta V(x) \rho_-(x + \tau)^2 e^{2kx} dx$$

$$I_2 = \text{const.}^{\rho+1} (1 + o(1)) e^{(\rho+1)k(\tau-t)} \int_{-1}^0 \delta \Gamma(x) \rho_-(x + \tau)^{\rho+1} e^{(\rho+1)kx} dx$$

## Example 3 – Dislocation ( $n = 1$ )

Check criterion for large  $t$ :

$$(\rho + 1) \underbrace{\int_{-\infty}^0 \delta V(x) w_0^2(x + \tau - t) dx}_{=: I_1} < 2 \underbrace{\int_{-\infty}^0 \delta \Gamma(x) |w_0(x + \tau - t)|^{\rho+1} dx}_{=: I_2}$$

Use asymptotics:

$$\frac{w_0(x)}{\rho_-(x) e^{kx}} \xrightarrow{x \rightarrow -\infty} \text{const.}, \quad \rho_{\pm} e^{\mp kx} = \text{Bloch mode for } -\frac{d^2}{dx^2} + V(x) - \lambda$$

$$I_1 = \text{const.}^2 (1 + o(1)) e^{2k(\tau-t)} \int_{-1}^0 \delta V(x) \rho_-(x + \tau)^2 e^{2kx} dx$$

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## Example 3 – Result

$$\int_{-1}^0 \left( V_0(x - \tau) - V_0(x + \tau) \right) p_-^2(x + \tau) e^{2\kappa x} dx < 0 \Rightarrow \exists \text{ ground state}$$

This can be used as follows:

- check numerically
- let spectral parameter  $\lambda \rightarrow -\infty$

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This can be used as follows:

- check numerically
- let spectral parameter  $\lambda \rightarrow -\infty$

### Theorem (Dohnal, Plum, R., 2010)

For  $\lambda \ll 0$  ground states for the dislocation problem exist if:

$$V_0(-\tau) \neq V_0(\tau) \quad \text{or} \quad V_0(-\tau) = V_0(\tau) \text{ and } V'_0(-\tau) > V'_0(\tau). \quad (1)$$

For  $|\tau|$  sufficiently small the above condition (1) holds if

$$V'_0(0) \neq 0 \quad \text{or} \quad V'_0(0) = 0 \text{ and } \text{sign } \tau V''_0(0) < 0. \quad (2)$$