Karlsruher Institut für Technologie

Preparatory Material
for the European Intensive Program in Bydgoszcz 2011

Analytical and computer assisted methods in mathematical models

September 4–18

Some theoretical background on Sobolev spaces and linear elliptic equations

Willy Dörfler

INTRODUCTION INTO
NUMERICAL METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS
• SOME THEORETICAL BACKGROUND
ON LINEAR ELLIPTIC EQUATIONS •

PROF. DR. W. DÖRFLER
INSTITUTE OF APPLIED AND NUMERICAL MATHEMATICS 2
KARLSRUHE INSTITUTE OF TECHNOLOGY

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Date: 13.05.2011.

1. Function spaces

$\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ will always denote a bounded *domain* (i.e., an open and connected set).

1.1. **Hölder spaces.**

The *(vector) space of continuous functions* is defined by

$$\mathcal{C}^0(\overline{\Omega}) \equiv \mathcal{C}^0(\overline{\Omega}, \mathbb{R}) := \Big\{ v : \overline{\Omega} \to \mathbb{R} \,:\, v \text{ is continuous} \Big\}.$$

For mappings $v : \overline{\Omega} \to \mathbb{R}^m$ with $m > 1$, we define $\mathcal{C}^0(\overline{\Omega})^m \equiv \mathcal{C}^0(\overline{\Omega}, \mathbb{R}^m)$ componentwise. As a norm on $\mathcal{C}^0(\overline{\Omega})^m$ we define

$$\|v\|_{\mathcal{C}^0(\overline{\Omega})^m} := \sup_{x \in \overline{\Omega}} \big\{ |v(x)| \big\},$$

where $|\,.\,|$ is a suitable vector norm. For $k \geq 1$ let

$$\mathcal{C}^k(\overline{\Omega}) := \Big\{ v \in \mathcal{C}^0(\overline{\Omega}) \,: v \text{ is } k\text{-times continuously differentiable in } \Omega \\ \text{and all } k\text{-th derivatives can be continuously extended to } \overline{\Omega} \Big\}.$$

Then, for $k \geq 0$, the normed spaces

$$\Big(\mathcal{C}^k(\overline{\Omega}),\ \|\,.\,\|_{\mathcal{C}^k(\overline{\Omega})} := \max_{0 \leq l \leq k} \big\{ \|\nabla^l(\,.\,)\|_{\mathcal{C}^0(\overline{\Omega})^{d^l}} \big\} \Big)$$

are *Banach*[1] *spaces* (complete normed spaces). For $\alpha \in (0,1]$ we define the semi-norm

$$\big[v\big]_{\alpha;\Omega} := \sup_{x,y \in \Omega \,:\, x \neq y} \left\{ \frac{|v(x) - v(y)|}{|x - y|^\alpha} \right\}$$

and the norm

$$\|v\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega})} := \|v\|_{\mathcal{C}^0(\overline{\Omega})} + \big[v\big]_{\alpha;\Omega}.$$

and let

$$\mathcal{C}^{0,\alpha}(\overline{\Omega}) := \Big\{ v \in \mathcal{C}^0(\overline{\Omega}) \,:\, \|v\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega})} < \infty \Big\}.$$

For $k \geq 1$ we then define

$$\mathcal{C}^{k,\alpha}(\overline{\Omega}) := \Big\{ v \in \mathcal{C}^k(\overline{\Omega}) \,:\, \text{all partial derivatives of order } k \text{ are in } \mathcal{C}^{0,\alpha}(\overline{\Omega}) \Big\}.$$

The normed spaces

$$\Big(\mathcal{C}^{k,\alpha}(\overline{\Omega}),\ \|\,.\,\|_{\mathcal{C}^{k,\alpha}(\overline{\Omega})} := \|\,.\,\|_{\mathcal{C}^k(\overline{\Omega})} + \big[\nabla^k(\,.\,)\big]_{\alpha;\Omega} \Big) \qquad (\textit{Hölder}^2 \textit{ space})$$

are Banach spaces. In case of infinitely many existing derivatives we define

$$\mathcal{C}^\infty(\Omega) := \Big\{ v : \Omega \to \mathbb{R} \,:\, v \text{ is infinitely often continuously differentiable in } \Omega \Big\}$$

and

$$\mathcal{C}^\infty(\overline{\Omega}) := \Big\{ v \in \mathcal{C}^\infty(\Omega) \,:\, v \text{ and all its derivatives can be continuously extended to } \overline{\Omega} \Big\}.$$

Both spaces are not normed [Rud78, Ch. 1.46].

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[1] Stefan Banach (1892–1945), Polish mathematician.
[2] Otto Hölder (1859–1937), German mathematician.

Examples.

(i) $\alpha = 1$: Functions in $\mathcal{C}^{0,1}(\overline{\Omega})$ are called *Lipschitz*[3] *continuous*. The function $x \mapsto |x|$ is in $\mathcal{C}^{0,1}([-1,1])$ but not in $\mathcal{C}^{1}([-1,1])$.

(ii) $d = 1$, $\Omega := (0,1)$, $v(x) := \sqrt{x}$: $v \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$ for $0 < \alpha \le 1/2$.

Remark. For $\alpha \in (0,1)$ we have the inclusions

$$\mathcal{C}^{1}(\overline{\Omega}) \subsetneq \mathcal{C}^{0,1}(\overline{\Omega}) \subsetneq \mathcal{C}^{0,\alpha}(\overline{\Omega}) \subsetneq \mathcal{C}^{0}(\overline{\Omega}).$$

$\mathcal{C}^{0,\alpha}(\overline{\Omega})$ is *compactly embedded* in $\mathcal{C}^{0}(\overline{\Omega})$ for $\alpha \in (0,1]$ by the Theorem of Arzelà–Ascoli[4] [DL00, Ch. II, §4], [Heu03, Satz 106.2] (i.e., the *embedding* $Id : \mathcal{C}^{0,\alpha}(\overline{\Omega}) \to \mathcal{C}^{0}(\overline{\Omega})$ is a compact mapping).

1.2. **Lebesgue and Sobolev spaces.** Here, we further assume that $\partial\Omega$ is Lipschitz continuous (or of class $\mathcal{C}^{0,1}$), i.e., $\partial\Omega$ can be locally represented as a graph of a Lipschitz continuous mapping $D \subset \mathbb{R}^{d-1} \to \partial\Omega$.

1.2.1. *Lebesgue spaces.* For $p \in [1,\infty]$ we define normed (vector) spaces by

$$\mathcal{L}^p(\Omega) := \Big\{ v : \Omega \to \mathbb{R} \,:\, v \text{ measurable and } \|v\|_{\mathcal{L}^p(\Omega)} < \infty \Big\},$$

where

$$\|v\|_{\mathcal{L}^p(\Omega)} := \begin{cases} \Big(\int_\Omega |v|^p \Big)^{1/p} & \text{for } p \in [1,\infty), \\ \operatorname*{ess\,sup}_{x\in\Omega} \big\{ |v(x)| \big\} & \text{for } p = \infty. \end{cases}$$

Functions in $\mathcal{L}^p(\Omega)$ are defined *pointwise a.e.* only. We have

$$v = w \text{ in } \mathcal{L}^p(\Omega) \quad :\Longleftrightarrow \quad (v-w)(x) = 0 \text{ pointwise a.e. in } \Omega.$$

The normed spaces

$$\Big(\mathcal{L}^p(\Omega), \|\,.\,\|_{\mathcal{L}^p(\Omega)} \Big) \qquad (\textit{Lebesgue}^5 \textit{ space})$$

are Banach spaces. For $p = 2$ we can define the scalar product (inner product)

$$(v,w)_{\mathcal{L}^2(\Omega)} := \int_\Omega vw$$

Thus $(v,v)_{\mathcal{L}^2(\Omega)} = \|v\|^2_{\mathcal{L}^2(\Omega)}$ and

$$\Big(\mathcal{L}^2(\Omega), (\,.\,,\,.\,)_{\mathcal{L}^2(\Omega)} \Big)$$

is a *Hilbert*[6] *space* (complete inner product space).

Example. Let $\Omega := B_1(0) \subset \mathbb{R}^d$ and $v(x) := |x|^\alpha$. Then

$$v \in \mathcal{L}^p(\Omega) \Longleftrightarrow \int_\Omega |v|^p < \infty \quad \Longleftrightarrow \quad \int_0^1 r^{\alpha p} r^{d-1}\, dr < \infty \quad \Longleftrightarrow \quad \int_0^1 r^{\alpha p + d - 1}\, dr < \infty$$
$$\Longleftrightarrow \quad \alpha p + d - 1 > -1 \quad \Longleftrightarrow \quad \alpha > -d/p.$$

[3] Rudolf Otto Sigismund Lipschitz (1832–1903), German mathematician.

[4] Giulio Ascoli (1843–1896), Cesare Arzelà (1847–1912), Italian mathematicians.

[5] Henri Léon Lebesgue (1875–1941), French mathematician.

[6] David Hilbert (1862–1943), German mathematician.

Remarks.

(i) A function $f \in \mathcal{L}^p(\Omega)$, $p \in [1,\infty]$, does not have well defined values on any smooth lower dimensional submanifold $S \subset \Omega$, since such a set is a zero set for the d-dimensional Lebesgue measure.

(ii) For bounded Ω we have $\mathcal{L}^p(\Omega) \subsetneq \mathcal{L}^q(\Omega)$ for $1 \le q < p \le \infty$. This follows from the estimate

$$\|f\|^q_{\mathcal{L}^q(\Omega)} = \int_\Omega |f|^q \le \Big(\int_\Omega 1\Big)^{1-1/s}\Big(\int_\Omega |f|^{qs}\Big)^{1/s}$$

for $s \ge 1$ (using Hölder's inequality) since taking $s = p/q$ yields $\|f\|_{\mathcal{L}^q(\Omega)} \le |\Omega|^{1/q-1/p}\|f\|_{\mathcal{L}^p(\Omega)}$. For unbounded Ω, there is no inclusion between $\mathcal{L}^p(\Omega)$ and $\mathcal{L}^q(\Omega)$ for $p \ne q$. However, it holds $\mathcal{L}^p(\Omega) \cap \mathcal{L}^\infty(\Omega) \subsetneq \mathcal{L}^q(\Omega) \cap \mathcal{L}^\infty(\Omega)$ for $1 \le p < q \le \infty$ (opposite to the above!). This is seen by proving $\|f\|_{\mathcal{L}^q(\Omega)} \le \|f\|^{1-p/q}_{\mathcal{L}^\infty(\Omega)}\|f\|^{p/q}_{\mathcal{L}^p(\Omega)}$ (show this for $g := f/\|f\|_{\mathcal{L}^\infty(\Omega)}$ exploiting $|g| \le 1$).

1.2.2. *Weak differentiability.* We weaken the concept of differentiability. Let

$$\mathcal{L}^1_{\text{loc}}(\Omega) := \Big\{ v : \Omega \to \mathbb{R} \,:\, v \in \mathcal{L}^1(K) \text{ for all } K \subset\subset \Omega \Big\}$$

and

$$\mathcal{C}^\infty_0(\Omega) := \Big\{ v \in \mathcal{C}^\infty(\overline{\Omega}) \,:\, \operatorname{supp}(v) \subset \Omega \Big\} \qquad (\textit{test function space}).$$

$v \in \mathcal{L}^1_{\text{loc}}(\Omega)$ is called *weakly differentiable*, if there exists a function $g \in \mathcal{L}^1_{\text{loc}}(\Omega)^d$ such that

$$\int_\Omega v \cdot \nabla w = -\int_\Omega g w \qquad \text{for all } w \in \mathcal{C}^\infty_0(\Omega).$$

g is then uniquely defined and for $v \in \mathcal{C}^1(K)$ one obtains $g = \nabla v$ on K for all open $K \subset \Omega$. We call g the *weak derivative* of v and write $g = \nabla v$ as in the classical case. Higher order weak derivatives are defined analogously.

Examples.

(i) $d = 1$, $\Omega := (-1,1)$, $v(x) := |x|$. What is v'? We claim:

$$v'(x) = \operatorname{sign}(x) := \begin{cases} -1 & \text{for } x < 0, \\ \ \ 0 & \text{for } x = 0, \\ \ \ 1 & \text{for } x > 0. \end{cases}$$

Indeed, applying the definition for arbitrary $w \in \mathcal{C}^\infty_0(\Omega)$ yields after integration by parts

$$\begin{aligned} \int_\Omega v w' &= \int_0^1 |x| w' = -\int_{-1}^0 x w' + \int_0^1 x w' = \int_{-1}^0 w - \int_0^1 w \\ &= -\Big(\int_{-1}^0 (-1) w + \int_0^1 (+1) w\Big) = -\int_0^1 \operatorname{sign}(x) w = -\int_\Omega v' w. \end{aligned}$$

(ii) Discontinuous functions on $\mathbb{R}$ do not have weak derivatives! Let $\Omega := (-1,1)$ and consider the function

$$\chi_{(0,\infty)}(x) := \begin{cases} 0 & \text{for } x \le 0, \\ 1 & \text{for } x > 0. \end{cases}$$

We show that $\chi_{(0,\infty)}$ does not have a weak derivative. If a weak derivative $g \in \mathcal{L}^1_{\text{loc}}(-1,1)$ would exist, it has to vanish a.e. and thus

$$0 = \int_{-1}^1 g w \overset{!}{=} -\int_{-1}^1 \chi_{(0,\infty)} w' = -\int_0^1 w' = w(0).$$

This, however, does not hold for *arbitrary* $w \in \mathcal{C}_0^\infty(-1,1)$.

Remark. The derivative of $\chi_{(0,\infty)}$ can be defined as a member of $\mathcal{C}^0(\overline{\Omega})'$, the *dual space* of $\mathcal{C}^0(\overline{\Omega})$. Thus it is a linear continuous mapping $\chi'_{(0,\infty)} : \mathcal{C}^0(\overline{\Omega}) \to \mathbb{R}$ satisfying

$$\chi'_{(0,\infty)}[v] = v(0) \qquad \text{for all } v \in \mathcal{C}^0(\overline{\Omega})$$

and therefore obeys the bound

$$\left|\chi'_{(0,\infty)}[v]\right| \le \|v\|_{\mathcal{C}^0(\Omega)}.$$

The mapping $\delta_0 := \chi'_{(0,\infty)}$ is called *Dirac*[7] *distribution in* 0.

1.2.3. *Sobolev spaces.* For $p \in [1,\infty]$ we define

$$\mathcal{W}^{1,p}(\Omega) := \Big\{ v \in \mathcal{L}^p(\Omega) \,:\, v \text{ is weakly differentiable and } |\nabla v| \in \mathcal{L}^p(\Omega) \Big\}.$$

For $m > 1$ correspondingly

$$\mathcal{W}^{m,p}(\Omega) := \Big\{ v \in \mathcal{W}^{m-1,p}(\Omega) \,:\, \text{All partial derivatives of order } m-1 \text{ are weakly differentiable and } |\nabla^m v| \in \mathcal{L}^p(\Omega) \Big\}.$$

For convenience we let $\mathcal{W}^{0,p}(\Omega) := \mathcal{L}^p(\Omega)$. With

$$\|v\|_{\mathcal{W}^{m,p}(\Omega)} := \Big| \big[\|\nabla^l v\|_{\mathcal{L}^p(\Omega)^{d^l}} \big]_l \Big|_{\ell^p}$$

the normed space

$$\Big(\mathcal{W}^{m,p}(\Omega), \|\,.\,\|_{\mathcal{W}^{m,p}(\Omega)} \Big) \qquad (\textit{Sobolev}^{8} \textit{ space})$$

is a Banach space. For $p = 2$ one can define the scalar product

$$\big(v, w\big)_{\mathcal{W}^{m,2}(\Omega)} := \sum_{l=0}^{m} \big(\nabla^l v, \nabla^l w\big)_{\mathcal{L}^2(\Omega)^{d^l}}$$

and obtain the Hilbert space

$$\Big(\mathcal{H}^m(\Omega), \big(\,.\,,\,.\,\big)_{\mathcal{H}^m(\Omega)} \Big) := \Big(\mathcal{W}^{m,2}(\Omega), \big(\,.\,,\,.\,\big)_{\mathcal{W}^{m,2}(\Omega)} \Big).$$

Clearly, $\mathcal{C}^m(\overline{\Omega}) \subset \mathcal{W}^{m,p}(\Omega)$ and $\mathcal{W}^{m,p}(\Omega) \subset \mathcal{W}^{n,p}(\Omega)$ for all $m > n$ and all $p \in [1,\infty]$. $\mathcal{W}^{m,p}(\Omega)$ can also be characterized by

$$\mathcal{W}^{m,p}(\Omega) = \overline{\mathcal{C}^\infty(\overline{\Omega})}^{\,\|\,.\,\|_{\mathcal{W}^{m,p}(\Omega)}}.$$

Especially, this means that $\mathcal{C}^\infty(\overline{\Omega})$ *is dense in* $\mathcal{W}^{m,p}(\Omega)$: For each $w \in \mathcal{W}^{m,p}(\Omega)$ there is a sequence $\{w_j\}_{j\in\mathbb{N}} \subset \mathcal{C}^\infty(\overline{\Omega})$ such that $w_j \to w$ with respect to $\|\,.\,\|_{\mathcal{W}^{m,p}(\Omega)}$ for $j \to \infty$.

[7] Paul Dirac (1902–1984), British physicist.
[8] Sergei Lwowitsch Sobolev (1908–1989), Russian mathematician.

Example. Let $\Omega := B_1(0) \subset \mathbb{R}^d$ and $v(x) := |x|^\alpha$. For which α holds $v \in \mathcal{W}^{m,p}(\Omega)$? By definition we have

$$\|v\|_{\mathcal{W}^{m,p}(\Omega)} < \infty \quad \Longleftrightarrow \quad \|\nabla^l v\|_{\mathcal{L}^p(\Omega)} < \infty \text{ for } l = 0, \dots, m.$$

Thus we obtain

$$\|\nabla^l v\|_{\mathcal{L}^p(\Omega)} \sim \int_0^1 |r^{\alpha-l}|^p r^{d-1}\, dr = \int_0^1 r^{(\alpha-l)p+d-1}\, dr \le \infty$$
$$\Longleftrightarrow \quad (\alpha - l)p + d - 1 > -1 \quad \Longleftrightarrow \quad \alpha > l - d/p.$$

In summary, $x \mapsto |x|^\alpha \in \mathcal{W}^{m,p}(\Omega) \Leftrightarrow \alpha > m - d/p$.

1.2.4. *Fractional Sobolev spaces.* For $s \in \mathbb{R}_{>0}$ let $\lfloor s \rfloor := \max\{m \in \mathbb{N} : m \le s\}$ and define the *fractional Sobolev spaces* (or *Sobolev–Slobodeckij spaces*) by

$$\mathcal{W}^{s,p}(\Omega) := \{w : \Omega \to \mathbb{R} : \|w\|_{\mathcal{W}^{s,p}(\Omega)} < \infty\},$$

where, with $m := \lfloor s \rfloor$,

$$\|w\|^p_{\mathcal{W}^{s,p}(\Omega)} := \|w\|^p_{\mathcal{W}^{m,p}(\Omega)} + \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x-y|^{(s-m+d/p)p}}\, dx\, dy.$$

This is, for $s \ge 0$ and $p \in (1, \infty)$, a separable and reflexive Banach space or a Hilbert space, if $p = 2$. $\mathcal{C}^\infty(\overline{\Omega})$ is dense in $\mathcal{W}^{s,p}(\Omega)$. $\mathcal{W}^{s,p}_0(\Omega)$ is defined to be the closure of $\mathcal{C}^\infty_0(\overline{\Omega})$ with respect to $\|\,.\,\|_{\mathcal{W}^{s,p}(\Omega)}$ and the vector valued function space is defined componentwise.
An alternative normequivalent construction for $p = 2$ is as follows: let E be a continuous *extension operator* $E : f : \Omega \to \mathbb{R} \;\mapsto\; Ef : \mathbb{R}^n \to \mathbb{R}$ (see e.g. [BS94, Ch. 14] [EG04, Ch. B.3.2]) and let

$$\|f\|^2_{\mathcal{H}^s(\Omega)} := \int_{\mathbb{R}^n} |\widehat{E}f(\xi)|^2 \big(1 + |\xi|^2\big)^s\, d\xi.$$

Details can be found in [Wlo82, §3] (or its english translation [Wlo87]). Such spaces are also defined for example in [BS94, Ch. 14] and [EG04, Ch. B.3.1].

Remark. The vector spaces presented in Sections 1.1 and 1.2 are of infinite dimension. We consider for example $\mathcal{L}^2(0,1)$. Then the set $\mathcal{B} := \{f_j\}_{j \in \mathbb{N}}$ defined by $f_j(x) := \cos(j\pi x)$ satisfies

$$\big(f_i, f_j\big)_{\mathcal{L}^2(\Omega)} = \frac{1}{2}\delta_{ij} \qquad \text{for all } i, j \in \mathbb{N}.$$

Thus, each finite subset of the unbounded set $\mathcal{B}$ is linearly independent.

1.3. **Properties of Sobolev functions.** In the following we assume that $\Omega \subset \mathbb{R}^d$ $(d \ge 1)$ is a bounded domain with $\partial\Omega$ assumed to be Lipschitz continuous.

1.3.1. *Embedding theorems.*

(i) Let $m, l \in \mathbb{N}$ and $p, q \in (1, \infty)$. If $m \ge l$ and $m - d/p > l - d/q$, then $\mathcal{W}^{m,p}(\Omega)$ in *continuously embedded* in $\mathcal{W}^{l,q}(\Omega)$, hence

$$\|v\|_{\mathcal{W}^{l,q}(\Omega)} \le C\|v\|_{\mathcal{W}^{m,p}(\Omega)} \qquad \text{for all } v \in \mathcal{W}^{m,p}(\Omega)$$

with a constant C independent of v. The embedding is compact if the first inequality is strict, $m > l$. $m - d/p$ is called the *Sobolev number* of $\mathcal{W}^{m,p}(\Omega)$ (see the example in Sect. 1.2.3).

(ii) If $m - d/p > k + \alpha$, then $\mathcal{W}^{m,p}(\Omega)$ is continuously (even compactly) embedded in $\mathcal{C}^{k,\alpha}(\overline{\Omega})$.

Example. Let $v \in \mathcal{W}^{m,p}(\Omega)$. Is v a bounded function, i.e., holds $v \in \mathcal{L}^\infty(\Omega)$? By (i), we have to verify $m - d/p > 0$. Let for example $m = 1$ and $p = 2$.

- $d = 1$: $m > 1/2$ is satisfied, hence $\|v\|_{\mathcal{L}^\infty(\Omega)} \leq C\|v\|_{\mathcal{W}^{1,2}(\Omega)}$.
- $d = 2$: $m > 1$ is not satisfied, hence there are functions in $\mathcal{W}^{1,2}(\Omega)$ that are not bounded! Indeed, let $\Omega := B_{1/e}(0) \subset \mathbb{R}^2$ and $v(x) := \log\big(\log(1/|x|)\big)$. Then $v \in \mathcal{W}_0^{1,2}(\Omega)$, but v is unbounded.
- $d = 3$: $m > 3/2$ is not satisfied, an unbounded function in $\mathcal{W}^{1,2}(\Omega)$ is $x \mapsto |x|^\alpha$ for $0 > \alpha > 1 - 3/2 = -1/2$.

1.3.2. *Traces.* Functions in $\mathcal{W}^{m,p}(\Omega)$ have "traces" on lower dimensional manifolds in the following sense: If $S \subset \Omega$ is a $d-r$–dimensional Lipschitz continuous hypersurface and $m > l$, $m - d/p > l - (d-r)/q$, then there exists a continuous linear mapping

$$\gamma_S : \mathcal{W}^{m,p}(\Omega) \to \mathcal{W}^{l,q}(S) \qquad (\textit{trace operator})$$

with $\gamma_S[v] = v\lfloor_S$ for all $v \in \mathcal{C}^\infty(\overline{\Omega})$ and

$$\|\gamma_S[v]\|_{\mathcal{W}^{l,q}(S)} \leq C\|v\|_{\mathcal{W}^{m,p}(\Omega)} \qquad \text{for all } v \in \mathcal{W}^{m,p}(\Omega)$$

with some constant C that is independent of v. In this sense, we will in the following use the notation $\gamma_S[v] = v\lfloor_S$ ("in trace sense") for all $v \in \mathcal{W}^{m,p}(\Omega)$. Note also, that $\gamma_{\partial\Omega}$ is surjective in this case: for each $g \in \mathcal{W}^{r,q}(\partial\Omega)$ there is a $v \in \mathcal{W}^{m,p}(\Omega)$ such that $\gamma_{\partial\Omega} v = g$.

Example. Let $v \in \mathcal{W}^{1,2}(\Omega)$. Then $v\lfloor_{\partial\Omega} \in \mathcal{L}^2(\partial\Omega)$ (in trace sense), since $1 - d/2 = 1/2 + (1-d)/2 > -(d-1)/2$.

1.3.3. *Poincaré–Friedrichs inequality.* Since functions in $\mathcal{W}^{1,p}(\Omega)$ admit boundary values in trace sense (see Sect. 1.3.2), we obtain a closed subspace of $\mathcal{W}^{1,p}(\Omega)$ by

$$\mathcal{W}_0^{1,p}(\Omega) := \ker(\gamma\lfloor_{\partial\Omega}) = \big\{v \in \mathcal{W}^{1,p}(\Omega) \,:\, v\lfloor_{\partial\Omega} = 0\big\}.$$

It can be equivalently characterised (on bounded domains) by

$$\mathcal{W}_0^{1,p}(\Omega) = \overline{\mathcal{C}_0^\infty(\Omega)}^{\,\|\,.\,\|_{\mathcal{W}^{1,p}(\Omega)}}.$$

For $p = 2$ we use also the notation $\mathcal{H}_0^1(\Omega) = \mathcal{W}_0^{1,2}(\Omega)$. The *Poincaré–Friedrichs*[9] *inequality* states that

$$\|v\|_{\mathcal{L}^p(\Omega)} \leq d_\Omega \|\nabla v\|_{\mathcal{L}^p(\Omega)} \qquad \text{for all } v \in \mathcal{W}_0^{1,p}(\Omega),$$

with d_Ω being the smallest diameter of Ω. As a result, $\|\nabla\,.\,\|_{\mathcal{L}^p(\Omega)}$ is a norm equivalent to $\|\,.\,\|_{\mathcal{W}^{1,p}(\Omega)}$, i.e., there is a constant $c > 0$ such that

$$c\|v\|_{\mathcal{W}^{1,p}(\Omega)} \leq \|\nabla v\|_{\mathcal{L}^p(\Omega)} \leq \|v\|_{\mathcal{W}^{1,p}(\Omega)} \qquad \text{for all } v \in \mathcal{W}_0^{1,p}(\Omega).$$

Obviously, we can take $c = 1 + d_\Omega$ here.

[9]Jules Henri Poincaré (1854–1912), French mathematician. Kurt Otto Friedrichs (1901–1982), German–American mathematician.

Remark.

(i) Let $p = 2$. The optimal constant in the Poincaré–Friedrichs inequality is given by $1/C_P$, where

$$C_P := \inf_{v \in \mathcal{H}^1_0(\Omega)} \left\{ \frac{\|\nabla v\|_{\mathcal{L}^2(\Omega)}}{\|v\|_{\mathcal{L}^2(\Omega)}} \right\} \qquad (\textit{Poincaré constant}).$$

One can show that this minimum is attained and that C_P is the *smallest eigenvalue* of $-\Delta u$ in Ω, i.e., the smallest (positive) number Λ such that the equation $-\Delta u = \Lambda u$ in Ω admits a non–trivial solution u with $u\lfloor_{\partial\Omega} = 0$.

(ii) Estimates of the same form as the Poincaré–Friedrichs inequality hold if $v\lfloor_S = 0$ is imposed for some measurable set $S \subset \partial\Omega$ with $\mathrm{meas}_{d-1}(S) \neq 0$. The constant, however, is then different and depends on S and Ω [Alt85, Ch. 5.15].

1.3.4. *Remark.* For $m > 0$ and $p \in (1, \infty)$, the dual space of $\mathcal{W}^{m,p}(\Omega)$ is $\big(\mathcal{W}^{m,p}(\Omega)\big)' = \mathcal{W}^{-m,p'}(\Omega)$ with $p' = p/(p-1)$.

Bibliographical notes. Definitions and properties of all the mentioned function spaces are contained in many modern books on theory and numerics of partial differential equations, for example [Alt85], [GT98], [BS94], [EG04], [GR05].

2. Linear elliptic equation

2.1. **Physical motivation.**

2.1.1. *Flux balance.* Let $\Omega \subset \mathbb{R}^d$, for $d \geq 1$, be a bounded domain. Inside Ω there is a *flux* $q : \Omega \to \mathbb{R}^d$ of a certain quantity, e.g., energy or mass. The total flow of q through a volume $V \subset \Omega$ is given by $\int_{\partial V} q \cdot n_{\partial V}$, where $n_{\partial V}$ is the outer normal field on ∂V. Inside V there might be a source or a sink for the flowing quantity of strength f. The total production (if positive) or loss (if negative) is $\int_V f$. Assuming conservation of the flowing quantity we require

$$\int_{\partial V} q \cdot n_{\partial V} = \int_V f$$

and by Gauß' theorem

$$\int_V \nabla \cdot q = \int_V f.$$

The quantity $\nabla \cdot q \equiv \mathrm{div}(q) := \sum_{i=1}^d \partial_i q_i$ is called "divergence". Since $V \subset \Omega$ is an arbitrary open set, we conclude

$$\nabla \cdot q = f \qquad \text{in } \Omega.$$

In case the flux q is driven by spatial differences between the flowing quantity u, one often assumes *Fourier's law*

$$q = -a\nabla u \qquad \text{in } \Omega.$$

Here, $a : \Omega \to \mathbb{R}_+$ is a material property describing the mobility of the flow. Thus we arrive at the *flux balance equation* or *diffusion equation*

$$-\nabla \cdot \big(a\nabla u\big) = f \qquad \text{in } \Omega. \tag{1}$$

Generalisations:

- Include transport by a flow field $b : \Omega \to \mathbb{R}^d$,
- include a linear reaction term: cu with $c : \Omega \to \mathbb{R}$,
- a may be a matrix field: $a : \Omega \to \mathbb{R}^{d,d}$ with $a(x)z \cdot z \geq 0$ for all $z \in \mathbb{R}^d$ and all $x \in \Omega$.

The general linear equation has then the form

$$-\nabla\cdot\big(a\nabla u\big) + b\cdot\nabla u + cu = f. \tag{1'}$$

2.1.2. *Boundary conditions.* The equation (1) or (1') needs to be accompanied by conditions on the boundary $\partial\Omega$. One might impose values for u on $\partial\Omega$ by a given function $g^{\mathrm{D}}:\partial\Omega\to\mathbb{R}$, hence

$$u = g^{\mathrm{D}} \qquad \text{on } \partial\Omega. \tag{2}$$

This is called *Dirichlet*[10] *boundary condition*. In case of a thermally isolated boundary, there will no flux through $\partial\Omega$, that is

$$n_{\partial\Omega}\cdot(a\nabla u) = 0 \qquad \text{on } \partial\Omega.$$

More generally, we may require

$$n_{\partial\Omega}\cdot(a\nabla u) = g^{\mathrm{N}} \qquad \text{on } \partial\Omega \tag{2'}$$

for a given function $g^{\mathrm{N}}:\partial\Omega\to\mathbb{R}$. (2') is called *Neumann*[11] *boundary condition*. Both conditions may appear at the same time on a *disjoint decomposition* of $\partial\Omega$,

$$\partial\Omega = \Gamma_{\mathrm{D}}\dot{\cup}\Gamma_{\mathrm{N}},$$

where we impose

$$\begin{aligned} u &= g^{\mathrm{D}} \qquad \text{on } \Gamma_{\mathrm{D}} \quad (\textit{Dirichlet boundary}),\\ n_{\partial\Omega}\cdot(a\nabla u) &= g^{\mathrm{N}} \qquad \text{on } \Gamma_{\mathrm{N}} \quad (\textit{Neumann boundary}). \end{aligned}$$

A pointwise mixture of these two conditions is called *Robin*[12] *boundary condition*

$$n_{\partial\Omega}\cdot(a\nabla u) + \widetilde{c}u = g^{\mathrm{R}} \qquad \text{on } \Gamma_{\mathrm{R}} \quad (\textit{Robin boundary})$$

with $\widetilde{c}, g^{\mathrm{R}}:\partial\Omega\to\mathbb{R}$.

2.2. **Classical theory.**

2.2.1. *Existence and regularity for the classical Dirichlet problem.* We consider, for given $\Omega, a, b, c, f, g^{\mathrm{D}}$, the boundary value problem (1'), (2). We require that

(i) Regularity assumptions. Assume that for some $\alpha\in(0,1)$
- Ω is bounded and $\partial\Omega$ is a local $\mathcal{C}^{2,\alpha}$–graph,
- $a\in\mathcal{C}^{1,\alpha}(\overline{\Omega})^{d,d}$, $b\in\mathcal{C}^{0,\alpha}(\overline{\Omega})^{d}$, $c,f\in\mathcal{C}^{0,\alpha}(\overline{\Omega})$,
- $g^{\mathrm{D}} = \widetilde{g}^{\mathrm{D}}\lfloor_{\partial\Omega}$ for some $\widetilde{g}^{\mathrm{D}}\in\mathcal{C}^{2,\alpha}(\overline{\Omega})$.

(ii) Structural conditions.
- $a:\Omega\to\mathbb{R}^{d,d}$ is symmetric and uniformly positive definit

  $$z\cdot a(x)z \geq a_0|z|^2 \qquad \text{for all } z\in\mathbb{R}^d \text{ and for all } x\in\Omega$$

  with some constant $a_0>0$ (a *uniformly elliptic*),
- $c\geq 0$.

Then there is a unique solution $u\in\mathcal{C}^{2,\alpha}(\overline{\Omega})$ of the boundary value problem (1'), (2) and this solution depends continuously on the data in the respective topology [GT98, Thm. 6.14]. Assuming, for $k\geq 0$, $\partial\Omega\in\mathcal{C}^{2+k,\alpha}$, $a\in\mathcal{C}^{1+k,\alpha}(\overline{\Omega})^{d,d}$, $b\in\mathcal{C}^{k,\alpha}(\overline{\Omega})^{d}$, $c,f\in\mathcal{C}^{k,\alpha}(\overline{\Omega})$, and $\widetilde{g}^{\mathrm{D}}\in\mathcal{C}^{2+k,\alpha}(\overline{\Omega})$ in the regularity requirements yields a solution $u\in\mathcal{C}^{2+k,\alpha}(\overline{\Omega})$ [GT98, Thm. 6.19].

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[10] Johann Peter Gustav Lejeune Dirichlet (1805–1859), German mathematician.

[11] János Neumann Margittai (John von Neumann) (1903–1957), Austrian–Hungarian – later American – mathematician.

[12] Victor Gustave Robin (1855–1897), French mathematician.

2.2.2. *Some counterexamples.*

(i) If $\partial\Omega$ is not differentiable, then u may not be in $\mathcal{C}^2(\overline{\Omega})$. Let $\Omega := (0,1)^2$, $f := 0$ and $g^{\mathrm{D}}(x_1, x_2) := x_1^2$. Then, necessarily, $\partial_1^2 u(x_1, 0) = 2$ and $\partial_2^2 u(0, x_2) = 0$ and therefore the second derivatives cannot be continuous in $(0,0)$.

(ii) Let $\Omega \subset \mathbb{R}^2$ be a domain with an *obtuse interior angle*, i.e., an interior angle of more than π. For example, let Ω be a segment of the unit disk with interior angle $\beta\pi$ for some $\beta \in (1,2)$ and choose $f := 0$ and $\widetilde{g}^{\mathrm{D}}(x) := r^{1/\beta}\sin(\phi/\beta)$ in polar coordinates (r,ϕ). The unique solution of this problem is $u(x) = \widetilde{g}^{\mathrm{D}}(x)$ for $x \in \Omega$. However, ∇u is unbounded in 0. It is said that u has a *corner singularity*. In three space dimensions there will be in addition also edge singularities. There exists a detailed theory that describes the behaviour of solutions near exceptional parts of $\partial\Omega$ for two and three space dimensions [Gri85] [KS87] [Dau88].

(iii) If Ω is unbounded, then a condition at "∞" is needed. A typical condition is $u(x) \to 0$ as $|x| \to \infty$. The existence and uniqueness theory is much more complicated than for bounded domains [MS60] [McO81].

(iv) If $f \in \mathcal{C}^0(\overline{\Omega})$, then u needs not to be in $\mathcal{C}^2(\overline{\Omega})$ and if $f \in \mathcal{C}^1(\overline{\Omega})$, then u needs not to be in $\mathcal{C}^{2,1}(\overline{\Omega})$ [GT98, Exercise 4.9].

(v) If a has zeros, there will be in general no solution at all or no solution in $\mathcal{W}_0^{1,2}(\Omega)$. As an example one may consider the equation $-(au')' = f$ with homogeneous boundary conditions for $a(x) := 0$ or $a(x) := x^m$ (for $m \in \mathbb{N}$) and $\Omega = (-1,1)$. For examples in two space dimensions see [GT98, Ch. 6.6].

(vi) In case of $c < 0$ uniqueness may not hold. Consider for example the *eigenvalue problem*

$$\begin{aligned} -\Delta u - \lambda u &= 0 \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

There is a monotone increasing sequence of positive numbers $\{\lambda_j\}_{j\in\mathbb{N}}$ (with $\lambda_j \to \infty$ for $j \to \infty$) for which non-zero solutions u exist (see Sect. **??**). For each such u we have the additional solutions su for all $s \in \mathbb{R}$.

(vii) If a is discontinuous, the problem cannot be stated in this way.

2.3. **Weak solutions.** Low regularity, as in the examples of Sect. 2.2.2, is a typical problem in applications. For example, a may be discontinuous or $\partial\Omega$ may have corners and edges (i.e., $\partial\Omega \notin \mathcal{C}^{2,\alpha}$). We thus need a concept to formulate the boundary value problem (1'), (2) for much weaker regularity conditions on the data. Of course, this will lead to a weaker solution concept.

2.3.1. *Weak formulation of the Dirichlet problem.* We multiply the equation (1') by an arbitrary $v \in \mathcal{C}_0^\infty(\Omega)$ and integrate by parts. Hence the solution u of (1'), (2) satisfies

$$\int_\Omega \Big\{ a\nabla u \cdot \nabla v + (b \cdot \nabla u)v + cuv \Big\} = \int_\Omega fv. \tag{3}$$

We now use (3) as a definition for a solution. With the notation

$$\begin{aligned} A[u,v] &:= \int_\Omega \Big\{ a\nabla u \cdot \nabla v + (b \cdot \nabla u)v + cuv \Big\}, \\ F[v] &:= \int_\Omega fv, \end{aligned}$$

the *weak formulation* of (1'), (2) reads: Seek $u \in \mathcal{W}^{1,2}(\Omega)$ with $u\lfloor_{\partial\Omega} = g^{\mathrm{D}}$ and

$$A[u,v] = F[v] \qquad \text{for all } v \in \mathcal{C}_0^\infty(\Omega). \tag{4}$$

This formulation allows for discontinuous a and for $\partial\Omega$ that is only Lipschitz continuous. The required regularity for u only needs one derivative in $\mathcal{L}^2(\Omega)$. Note that these functions may not even continuous (see Sect. 2.2.2). The problem (1') can be solved in a very general setting.

2.3.2. ***The Lax–Milgram Theorem.*** Let $\mathcal{H}$ be a Hilbert space with norm $\|\,.\,\|_{\mathcal{H}}$ and let $B : \mathcal{H}\times\mathcal{H} \to \mathbb{R}$ be a *coercive* and *continuous* bilinear form, that is, there are strictly positive constants α_0, α_1 such that

$$B[v,v] \geq \alpha_0 \|v\|_{\mathcal{H}}^2,$$
$$\big|B[v,w]\big| \leq \alpha_1 \|v\|_{\mathcal{H}} \|w\|_{\mathcal{H}} \qquad \text{for all } v,w \in \mathcal{H},$$

and let $G : \mathcal{H} \to \mathbb{R}$ be a *continuous linear functional* ($:\Leftrightarrow G \in \mathcal{H}'$), i.e.,

$$\big|G[v]\big| \leq \|G\|_{\mathcal{H}'} \|v\| \qquad \text{for all } v \in \mathcal{H}.$$

Then there exists a unique solution $u \in \mathcal{H}$ of the *variational equality*

$$B[u,v] = G[v] \qquad \text{for all } v \in \mathcal{H},$$

that obeys the bound (*a priori estimate*)

$$\|u\|_{\mathcal{H}} \leq \frac{1}{\alpha_0}\|G\|_{\mathcal{H}'}.$$

Proof. This theorem is a standard result from functional analysis, see for example [Alt85, Ch. 4.9]. The estimate follows easily from

$$\|u\|_{\mathcal{H}}^2 \leq \frac{1}{\alpha_0} B[u,u] = \frac{1}{\alpha_0} G[u] \leq \frac{1}{\alpha_0}\|G\|_{\mathcal{H}'}\|u\|_{\mathcal{H}}.$$

□

2.3.3. *Existence and regularity for the weak Dirichlet problem.*

(i) Let $a \in \mathcal{L}^\infty(\Omega)^{d,d}$, $b \in \mathcal{W}^{1,\infty}(\Omega)^d$, $c \in \mathcal{L}^\infty(\Omega)$, $\Omega \subset \mathbb{R}^d$ a bounded domain with Lipschitz continuous boundary, and $F[v] := \int_\Omega fv$ for some $f \in \mathcal{L}^2(\Omega)$. Let g^{D} be the trace of a function $\widetilde{g}^{\mathrm{D}} \in \mathcal{W}^{1,2}(\Omega)$. Assume that a is symmetric positive definit a.e. as required in Sect. 2.2.1 und that $c - 1/2\nabla\cdot b \geq 0$ a.e. in Ω. Then there exists a unique solution $u \in \mathcal{W}^{1,2}(\Omega)$ of the variational problem (3) und $\|u\|_{\mathcal{W}^{1,2}(\Omega)} \leq C$ with C depending on the data.

(ii) Let, in addition to the requirements stated in (i), $a \in \mathcal{C}^{0,1}(\Omega)^{d,d}$ and $\widetilde{g}^{\mathrm{D}} \in \mathcal{W}^{2,2}(\Omega)$. Then $u \in \mathcal{W}^{2,2}(D)$ for every $D \subset\subset \Omega$. In case $D \subset \Omega$ with $S := D \cap \partial\Omega \neq \emptyset$, this holds true if S is of class $\mathcal{C}^{2,\alpha}$. If $\partial\Omega$ is $\mathcal{C}^2$–regular as a whole or Ω is convex, then $u \in \mathcal{W}^{2,2}(\Omega)$. Assuming, for $k \geq 0$, $\partial\Omega \in \mathcal{C}^{2+k}$, $a \in \mathcal{W}^{k,\infty}(\Omega)^{d,d}$, $b \in \mathcal{W}^{1+k,\infty}(\Omega)^d$, $c, f \in \mathcal{W}^{k,\infty}(\Omega)$, and $\widetilde{g}^{\mathrm{D}} \in \mathcal{W}^{2+k,2}(\Omega)$ in the regularity requirements yields a solution $u \in \mathcal{W}^{2+k,2}(\Omega)$ [GT98, Thm. 8.12, 8.13].

Proof. Proof of (i): We want to apply Theorem 2.3.2. First split $u = u_0 + \widetilde{g}^{\mathrm{D}}$ with $u_0 \in \mathcal{W}_0^{1,2}(\Omega)$. u_0 will then be the solution of

$$A[u_0,v] = \widetilde{F}[v] := F[v] - A(\widetilde{g}^{\mathrm{D}}, v) \qquad \text{for all } v \in \mathcal{C}_0^\infty(\Omega).$$

By density of $\mathcal{C}_0^\infty(\Omega)$ in $\mathcal{W}_0^{1,2}(\Omega)$, we can state the problem equivalently as: Seek $u_0 \in \mathcal{W}_0^{1,2}(\Omega)$ such that $A[u_0,v] = \widetilde{F}[v]$ for all $v \in \mathcal{W}_0^{1,2}(\Omega)$. Now let $\mathcal{H} := \mathcal{W}_0^{1,2}(\Omega)$ and $\|\,.\,\|_{\mathcal{H}} := \|\nabla\,.\,\|_{\mathcal{L}^2(\Omega)}$. Then using the Cauchy–Schwarz inequality and the Poincaré–Friedrichs inequality we have for all $v,w \in \mathcal{H}$

$$\begin{aligned}
\big|A[v,w]\big| &\leq \|a\|_{\mathcal{L}^\infty(\Omega)}\|v\|_{\mathcal{H}}\|w\|_{\mathcal{H}} + \|b\|_{\mathcal{L}^\infty(\Omega)}\|v\|_{\mathcal{H}}\|w\|_{\mathcal{L}^2(\Omega)} + \|c\|_{\mathcal{L}^\infty(\Omega)}\|v\|_{\mathcal{L}^2(\Omega)}\|w\|_{\mathcal{L}^2(\Omega)} \\
&\leq \Big(\|a\|_{\mathcal{L}^\infty(\Omega)} + d_\Omega\|b\|_{\mathcal{L}^\infty(\Omega)} + d_\Omega^2\|c\|_{\mathcal{L}^\infty(\Omega)}\Big)\|v\|_{\mathcal{H}}\|w\|_{\mathcal{H}} \\
&\leq C(A,b,c,\Omega)\,\|v\|_{\mathcal{H}}\|w\|_{\mathcal{H}}
\end{aligned}$$

and

$$\begin{aligned} A[v,v] &= \int_\Omega \Big\{ a\nabla v\cdot\nabla v + (b\cdot\nabla v)v + c|v|^2 \Big\} \\ &\geq \int_\Omega \Big\{ a_0|\nabla v|^2 + (c - \frac{1}{2}\nabla\cdot b)|v|^2 \Big\} \geq a_0 \int_\Omega |\nabla v|^2 = a_0\|v\|^2_{\mathcal{H}}. \end{aligned}$$

For $\widetilde{F}$ we immediately get

$$\widetilde{F}[v] \leq \Big(\|F\|_{\mathcal{W}^{-1,2}(\Omega)} + C(A,b,c,\Omega)\|g^{\mathrm{D}}\|_{\mathcal{W}^{1,2}(\Omega)} \Big) \|v\|_{\mathcal{H}}.$$

By the Lax–Milgram theorem there exists a unique $u_0 \in \mathcal{H}$ such that $A[u_0,v] = \widetilde{F}[v]$ for all $v \in \mathcal{H}$ and the bound for u_0 follows with the previous estimates. The result for u is then immediate. □

2.3.4. *Examples.*

(i) The theoretical result allows the right hand side F to be a functional. For $\Omega := (-1,1) \subset \mathbb{R}$ one may take for example δ_0, the Dirac distribution from Sect. 1.2.2, since such F is continuous by Sect. 1.3.1: $\big|F[v]\big| = \big|\delta_0[v]\big| = |v(0)| \leq \|v\|_{\mathcal{L}^\infty(\Omega)} \leq C\|v\|_{\mathcal{W}^{1,2}_0(\Omega)}$. This is not true for $d \geq 2$ [ibd]. Examples on $\mathbb{R}^d$ could be $F[v] := \int_\Omega f\cdot\nabla v$ for some $f \in \mathcal{L}^2(\Omega)^d$, or, for a smooth $d-1$ dimensional manifold S in Ω, $F[v] := \int_S v$. The continuity of F in the second example follows from the trace theorem 1.3.2.

(ii) This solution concept works for solutions with singularities as in Sect. 2.2.2. They are all in $\mathcal{W}^{1,2}(\Omega)$, but not in $\mathcal{W}^{2,2}(\Omega)$. However, they are in $\mathcal{W}^{2,p}(\Omega)$ for some $p \in (1,2)$. Let for example $\Omega := (-1,1)^2 \subset \mathbb{R}^2$ and $a = 1$ in $\{[x_1,x_2] \in \Omega : x_1x_2 > 0\}$, $a = 100$ in $\{[x_1,x_2] \in \Omega : x_1x_2 < 0\}$, $b = 0$, $c = 0$, $f = 1$, $g^{\mathrm{D}} = 0$. A unique solution exists in $\mathcal{W}^{1,2}(\Omega)$ and it can be shown that u behaves like r^β with $\beta \approx 0.1$ for $r \to 0$. Thus $u \in \mathcal{W}^{2,p}(\Omega)$ for $p \in (1, 2/(2-\beta))$ or $u \in \mathcal{W}^{1+\beta,2}(\Omega)$ only [Kel75] [MNS00, Sect. 5.3].

2.3.5. *The Neumann problem.* Let for simplicity $b = c = 0$. Assume that $\partial\Omega = \Gamma_{\mathrm{D}}\dot\cup\Gamma_{\mathrm{N}}$ (disjoint decomposition) and that the boundary condition are as follows

$$\begin{aligned} u &= g^{\mathrm{D}} \qquad \text{on } \Gamma_{\mathrm{D}}, \\ n_{\partial\Omega}\cdot\big(a\nabla u\big) &= g^{\mathrm{N}} \qquad \text{on } \Gamma_{\mathrm{N}}. \end{aligned}$$

By subtracting a global function $\widetilde{g}^{\mathrm{D}}$, as in the proof of Sect. 2.3.3, we may assume that $g^{\mathrm{D}} = 0$. We now multiply (1) by an arbitrary function $v \in \mathcal{C}^\infty(\overline{\Omega})$ with $v(x) = 0$ for all $x \in \Gamma_{\mathrm{D}}$ and obtain

$$\begin{aligned} A[u,v] := \int_\Omega a\nabla u\cdot\nabla v &= \int_\Omega fv + \int_{\partial\Omega} n_{\partial\Omega}\cdot(a\nabla u)v \\ &= \int_\Omega fv + \int_{\Gamma_{\mathrm{N}}} g^{\mathrm{N}} v =: F[v]. \end{aligned}$$

Using this as a definition of a weak solution $u \in \mathcal{H} := \big\{w \in \mathcal{W}^{1,2}(\Omega) : w\lfloor_{\Gamma_{\mathrm{D}}} = 0\big\}$ with $\|\,.\,\|_{\mathcal{H}} := \|\nabla\,.\,\|_{\mathcal{L}^2(\Omega)}$, we can proceed as in Sect. 2.3.3.(i). Assuming $g^{\mathrm{N}} \in \mathcal{L}^2(\Gamma_{\mathrm{N}})$ we obtain with Sect. 1.3.2

$$\begin{aligned} A[v,v] &\geq \alpha_0\|v\|^2_{\mathcal{H}}, \\ \big|F[v]\big| = \Big|\int_\Omega fv + \int_{\Gamma_{\mathrm{N}}} g^{\mathrm{N}} v\Big| &\leq \|f\|_{\mathcal{L}^2(\Omega)}\|v\|_{\mathcal{W}^{1,2}(\Omega)} + \|g^{\mathrm{N}}\|_{\mathcal{L}^2(\Gamma_{\mathrm{N}})}\|v\|_{\mathcal{L}^2(\partial\Omega)} \\ &\leq C(f,g^{\mathrm{N}})\,\|v\|_{\mathcal{W}^{1,2}(\Omega)}. \end{aligned}$$

In case $\mathrm{meas}_{d-1}(\Gamma_{\mathrm{D}}) \neq 0$, the Poincaré–Friedrichs inequality 1.3.3 still holds and F is proved to be continuous on $\mathcal{H}$. However, in case $\mathrm{meas}_{d-1}(\Gamma_{\mathrm{D}}) = 0$ this is not true, and in fact, the problem might not be solvable: if u is a solution, then $u+k$ is also a solution for any $k \in \mathbb{R}$. However, if we impose the necessary condition $\int_\Omega f + \int_{\Gamma_N} g^N = 0$ and require the additional constraint $\int_\Omega u = 0$ (to fix one specific k) one can prove again existence and uniqueness. To this end let

$$\mathcal{H} := \Big\{ v \in \mathcal{W}^{1,2}(\Omega) \,:\, \int_\Omega v = 0 \Big\}$$

and let $\|\cdot\|_{\mathcal{H}} := \|\nabla\,.\,\|_{\mathcal{L}^2(\Omega)}$. One can prove the Poincaré–type inequality

$$\|v\|_{\mathcal{L}^2(\Omega)} \leq C(\Omega)\|v\|_{\mathcal{H}} \qquad \text{for all } v \in \mathcal{H}$$

and from this the norm equivalence between $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{W}^{1,2}(\Omega)}$ as in Sect. 1.3.3. Now we define the $\mathcal{L}^2$–orthogonal decomposition

$$\begin{aligned} \mathcal{W}^{1,2}(\Omega) &= \mathcal{H} \oplus \mathbb{R} \\ v &= w + m(v) \end{aligned}$$

with

$$\begin{aligned} m(v) &:= \frac{1}{|\Omega|}\int_\Omega v \;\in \mathbb{R}, \\ w &:= v - m(v) \in \mathcal{H}. \end{aligned}$$

The weak equation can now be formulated in $\mathcal{H}$. For all $v \in \mathcal{W}^{1,2}(\Omega)$ one has

$$\begin{aligned} \int_\Omega \nabla u \cdot \nabla v &= \int_\Omega fv + \int_{\Gamma_N} g^N v \\ &= \int_\Omega fw + \int_{\Gamma_N} g^N w + \Big(\int_\Omega f + \int_{\Gamma_N} g^N\Big) m(v) \\ &= \int_\Omega fw + \int_{\Gamma_N} g^N w. \end{aligned}$$

Since $w \in \mathcal{H}$ is also arbitrary, we will seek $u \in \mathcal{H}$ such that

$$\int_\Omega \nabla u \cdot \nabla w = \int_\Omega fw + \int_{\Gamma_N} g^N w \qquad \text{for all } w \in \mathcal{H}.$$

Existence and uniqueness for this problem is now proved as in Sect. 2.3.3.

Remark. If we would have defined $\|\,.\,\|_{\mathcal{H}} := \|\,.\,\|_{\mathcal{W}^{1,2}(\Omega)}$, then F would immediately be continuous but then we would need the same requirements as above to show that A is coercive on $\mathcal{H}$.

2.3.6. *The Dirichlet problem on an exterior domain.* Let $G \subset \mathbb{R}^d$ be a bounded and simply connected domain with Lipschitz continuous boundary and define $\Omega := \mathbb{R}^d \setminus G$. Such an Ω is called *exterior domain.* We further assume $a = 1$, $b = 0$, and $c = 0$ and assume homogeneous boundary conditions on ∂G and for $|x| \to \infty$. To this end we seek the solution in (compare Sect. 1.3.3)

$$\mathcal{W}_0^{1,p}(\Omega) := \overline{\mathcal{C}_0^\infty(\Omega)}^{\,\|\nabla\,\cdot\,\|_{\mathcal{L}^2(\Omega)}}.$$

On an exterior domain the Poincaré–Friedrichs inequality does not hold so we need different arguments to prove continuity of $f \mapsto \int_\Omega fv$. The only continuous embedding $\mathcal{W}_0^{1,2}(\Omega) \to \mathcal{L}^p(\Omega)$ holds in case of equality of the Sobolev numbers, $1 - d/2 \overset{!}{=} -d/p \Leftrightarrow d > 2$ and $p = 1/(1/2 - 1/d)$ [BF96]. If we fix $d = 3$, then $p = 6$ and we find (with Hölder's inequality)

$$\Big|\int_\Omega fv\Big| \leq \Big(\int_\Omega |f|^{6/5}\Big)^{5/6}\Big(\int_\Omega |v|^6\Big)^{1/6} \leq \|f\|_{\mathcal{L}^{6/5}(\Omega)}\|\nabla v\|_{\mathcal{L}^2(\Omega)}.$$

Alternatively, one may exploit *Hardy's*[13] *inequality* [BGH90] [SS96] (assuming that without loss of generality $0 \in G$)

$$\int_\Omega \frac{|v|^2}{|x|^2} \le 4 \int_\Omega |\nabla v|^2.$$

We then get

$$\Big| \int_\Omega fv \Big| \le \int_\Omega |x| f \frac{|v|}{|x|} \le \Big(\int_\Omega |x|^2 |f|^2 \Big)^{1/2} \Big(\int_\Omega \frac{|v|^2}{|x|^2} \Big)^{1/2} \le 2 \| \, |x| f \|_{\mathcal{L}^2(\Omega)} \| \nabla v \|_{\mathcal{L}^2(\Omega)}.$$

Thus we need $f \in \mathcal{L}^{6/5}(\Omega)$ or $|x| f \in \mathcal{L}^2(\Omega)$. Note that $u \in \mathcal{L}^2(\Omega)$ does in general not hold. More general results on this topic can be found in the monograph [SS96].

2.3.7. *Dirichlet principle.* Let a, c, f, Ω as in Sect. 2.3.3.(i) but with $b = 0$. Then u is a solution (3) if and only if u is a minimiser of

$$I : \mathcal{W}_0^{1,2}(\Omega) \to \mathbb{R}$$
$$I(w) := \int_\Omega \Big\{ \frac{1}{2} a \nabla w \cdot \nabla w + \frac{1}{2} c |w|^2 - f w \Big\}.$$

Proof. It is easy to see that $I \in \mathcal{C}^2\big(\mathcal{W}_0^{1,2}(\Omega)\big)$ is strictly convex and bounded from below. Hence, if u is a minimiser of I, it is completely characterised by $I'(u)[v] = 0$ for all $v \in \mathcal{W}_0^{1,2}(\Omega)$ and this is (3). ☐

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