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Analytical and computer assisted methods in mathematical models

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Basics from functional analysis and Hilbert spaces

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Ref.: W. Rudin, Real and Complex Analysis, 3^{rd} Ed., McGraw-Hill, 1987, Chapter 4.

Definition 1 (Bounded, compact linear operators) *Let $(X, \|\cdot\|)$ and $(Y, |||\cdot|||)$ be normed spaces. A linear operator $A : X \to Y$ is called*

(i) bounded, if

$$\|A\| := \sup_{x \in X \setminus \{0\}} \frac{|||Ax|||}{\|x\|} < \infty.$$

In the case $Y = \mathbb{R}$ bounded linear operators are called bounded linear **functionals**.

(ii) compact, if for every bounded sequence $(x_k)_{k\in\mathbb{N}}$ in X the sequence $(Ax_k)_{k\in\mathbb{N}}$ in Y has a convergent subsequence.

Definition 2 (Hilbert space) *Let H be a real vector space with inner product $\langle \cdot, \cdot \rangle$, i.e.*

(i) $\langle x, y \rangle = \langle y, x \rangle$,

(ii) $\langle x, x \rangle \geq 0$ *and* $= 0$ *if and only if* $x = 0$,

(iii) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$.

Then

$$\|x\| := \sqrt{\langle x, x \rangle}$$

defines a norm on H. If H equipped with the above norm is complete, then $(H, \langle \cdot, \cdot \rangle)$ is called a Hilbert space.

Definition 3 (Orthogonality) *Let $\langle \cdot, \cdot \rangle$ be an inner product on the real vector space H and let $V \subset H$.*

(i) $x \perp y :\Leftrightarrow \langle x, y \rangle = 0$,

(ii) $x \perp V :\Leftrightarrow x \perp v$ *for all* $v \in V$,

(iii) $V^\perp := \{x \in H : x \perp V\}$.

Theorem 4 (Distance minimizer) *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let $V \subset H$ be a closed subspace. Then*

(i) $\forall x \in H$ there exists a unique $v_0 \in V$ such that

$$\|x - v_0\| = \operatorname{dist}(x, V) = \inf_{v \in V} \|x - v\|.$$

Moreover $x - v_0 \perp V$.

(ii) $H = V \oplus V^\perp$.

Theorem 5 (Riesz representation theorem) *Let $(H, \langle \cdot, \cdot \rangle)$ be a real Hilbert space and let $\phi : H \to \mathbb{R}$ be a bounded linear functional. Then there exists a unique $u \in H$ such that*

$$\phi(x) = \langle u, x \rangle \text{ for all } x \in H.$$

Note: $\|\phi\| = \|u\|$.

Definition 6 (Weak convergence) *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a real Hilbert space. A sequence* $(x_k)_{k\in\mathbb{N}}$ *in* H *is called weakly convergent to* $x \in H$ *if*

$$\lim_{k\to\infty} \langle x_k, y\rangle = \langle x, y\rangle \text{ for all } y \in H.$$

One writes $x_k \rightharpoonup x$ *as* $k \to \infty$ *for* **weakly convergent** *sequences. In contrast, one writes* $x_k \to x$ *as* $k \to \infty$ *for* **strongly convergent** *(norm convergent) sequences, i.e. if* $\|x_k - x\| \to 0$ *as* $k \to \infty$.

Lemma 7 (Relation between weak and strong convergence) *Let* $(x_k)_{k\in\mathbb{N}}$ *be a sequence in a Hilbert space* H *and let* $x \in H$.

(i) If $x_k \to x$ *as* $k \to \infty$ *then* $x_k \rightharpoonup x$ *as* $k \to \infty$*. In infinite dimensions the reverse is in general false.*

(ii) The following equivalence holds:

$$x_k \to x \text{ as } k \to \infty \iff x_k \rightharpoonup x \text{ and } \|x_k\| \to \|x\| \text{ as } k \to \infty.$$

Theorem 8 (Banach-Alaoglu for Hilbert spaces) *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a real Hilbert space and let* $(x_k)_{k\in\mathbb{N}}$ *be a sequence in* H.

(i) If $(x_k)_{k\in\mathbb{N}}$ *weakly converges to* $x \in H$ *then the sequence* $(\|x_k\|)_{k\in\mathbb{N}}$ *is bounded and*

$$\|x\| \le \liminf \|x_k\|.$$

(ii) If $(x_k)_{k\in\mathbb{N}}$ *is bounded then there exists a weakly convergent subsequence.*

Definition 9 (Separable space) *A Banach space* $(X, \|\cdot\|)$ *is called separable, if there exists a countable set* $Z = \{z_1, z_2, z_3, \dots\}$, $Z \subset X$ *such that* $\overline{Z} = X$.

Examples: $L^p(\Omega), W^{k,p}(\Omega)$ are separable if $1 \le p < \infty$. $L^\infty(\Omega), W^{k,\infty}(\Omega)$ are not separable if Ω is open and $\Omega \ne \emptyset$.

Definition 10 (Orthonormal system, orthonormal basis) *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a real, infinite-dimensional Hilbert space. A set* $B = \{u_i : i \in \mathbb{N}\} \subset H$ *is called an orthonormal system if*

$$\langle u_i, u_j\rangle = \delta_{ij}.$$

If additionally,

$$u = \sum_{i=1}^{\infty} \langle u, u_i\rangle u_i \quad \text{for all } u \in H,$$

then B *is called an orthonormal basis.*

Theorem 11 (Existence of orthonormal basis) *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a real, separable Hilbert space. Then* H *has an orthonormal basis.*

Theorem 12 (Convergence of the abstact Fourier series) *Let $(H, \langle \cdot, \cdot \rangle)$ be a real Hilbert space and let $B = \{u_i : i \in \mathbb{N}\}$ be an orthonormal system. For every $u \in H$ the series*

$$\hat{u} := \sum_{i=1}^{\infty} \langle u, u_i \rangle u_i \quad \textit{(abstract Fourier series)}$$

is convergent.

Proof: Fix $u \in H$. For $k \in \mathbb{N}$ let $\hat{u}_k := \sum_{i=1}^{k} \langle u, u_i \rangle u_i$. Then $\|\hat{u}_k\|^2 = \sum_{i=1}^{k} |\langle u, u_i \rangle|^2$ and

Bessel's equation: $$\|u - \hat{u}_k\|^2 = \|u\|^2 - \sum_{i=1}^{k} |\langle u, u_i \rangle|^2 = \|u\|^2 - \|\hat{u}_k\|^2.$$

Hence

Bessel's inequality: $$\sum_{i=1}^{\infty} |\langle u, u_i \rangle|^2 \leq \|u\|^2.$$

For $k > l$

$$\|\hat{u}_k - \hat{u}_l\|^2 = \| \sum_{i=l+1}^{k} \langle u, u_i \rangle u_i \|^2 = \sum_{i=l+1}^{k} |\langle u, u_i \rangle|^2 < \epsilon \text{ for } k > l \geq l_0(\epsilon)$$

due to Bessel's inequality. Hence $(\hat{u}_k)_{k \in \mathbb{N}}$ is a Cauchy sequence and thus convergent. $\square$