On the operator equation $e^A = e^B$

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Abstract

Suppose that $A$ and $B$ are bounded linear operators on a complex Hilbert space and that $e^A = e^B$. It is well-known that if the spectrum of $A$ is incongruent (mod 2πi) then $AB = BA$. In this note we show that if $A$ is normal and $\|A\| \leq \pi$ then $e^A = e^B$ implies that $A^2B = BA^2$. If $B$ is also normal, $\|B\| \leq \pi$ and $-i\pi$ is not an eigenvalue of $A$ then we show that $e^A = e^B$ implies $AB = BA$ and $(A - B)^2 = 2\pi i(A - B)$.

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Throughout this paper let $\mathcal{H}$ denote a complex Hilbert space and $\mathcal{L}(\mathcal{H})$ the Banach algebra of all bounded linear operators on $\mathcal{H}$. For $A \in \mathcal{L}(\mathcal{H})$ the spectrum, the set of eigenvalues, and the spectral radius of $A$ are denoted by $\sigma(A)$, $\sigma_p(A)$, and $r(A)$, respectively. For the resolvent set of $A$ we write $\rho(A)$. We use $N(A)$ and $A(\mathcal{H})$ to denote the kernel and the range of $A$, respectively.

We say that $\sigma(A)$ is incongruent (mod 2πi), if

$$\sigma(A) \cap \sigma(A + 2j\pi i) = \emptyset \quad \text{for} \ j = \pm 1, \pm 2, \ldots.$$ 

If $A \in \mathcal{L}(\mathcal{H})$ is normal (AA* = A*A) and has the spectral resolution

$$A = \int_{\sigma(A)} \lambda dE(\lambda),$$

(1)
let $E(\Omega)$ denote the associated projection measure defined on the Borel subsets $\Omega \subseteq \sigma(A)$. It is convenient to think of $E$ as being defined for all Borel subsets in $C$: put $E(\Omega) = E(\Omega \cap \sigma(A))$. It is well-known that $\|A\| = r(A)$ if $A$ is normal.

Let $T \in L(H)$. The map $\delta_T : L(H) \to L(H)$, defined by

$$\delta_T(C) = CT - TC \quad (C \in L(H))$$

is called the inner derivation determined by $T$. $\delta_T$ is a bounded linear operator on $L(H)$ with $\|\delta_T\| \leq 2\|T\|$.

It is shown in [3] that

$$\sigma(\delta_T) = \{\lambda - \mu : \lambda, \mu \in \sigma(T)\}, \quad (2)$$

From [4, Proposition 6.4.8] it follows that $e^{\delta_T}(C) = e^{T}Ce^{-T}$ for all $C \in L(H)$.

Throughout this note let $f$ denote the entire function $f : C \to C$ given by

$$f(z) = \begin{cases} z^{-1}(e^z - 1), & \text{if } z \neq 0, \\ 1, & \text{if } z = 0. \end{cases}$$

From $zf(z) = f(z)z = e^z - 1$ and (3) we get

$$f(\delta_T)(\delta_T(C)) = e^{-T}Ce^T - C \quad \text{for all } C \in L(H). \quad (4)$$

For a bounded linear operator $F$ on $L(H)$ we denote the kernel of $F$ by $N(F)$. 

**Proposition 1.** Let $A \in L(H)$ such that $r(A) \leq \pi$. Let

$$M_A = \{\lambda \in \sigma(\delta_A) : f(\lambda) = 0\}.$$

(a) If $\sigma(A)$ is incongruent (mod $2\pi i$), then $M_A = \emptyset$.
(b) If $M_A = \emptyset$, then $f(\delta_A)$ is an invertible operator on $L(H)$.
(c) $M_A \subseteq [2\pi i, -2\pi i]$.
(d) $N(f(\delta_A)) = N(\delta_A - 2\pi i) \oplus N(\delta_A + 2\pi i)$.
(e) If $C \in N(\delta_A - 2\pi i)$ or $C \in N(\delta_A + 2\pi i)$, then $C^2 = 0$.

**Proof.** (a) and (b) are clear.

(c) Take $\lambda \in M_A$, then $0 \neq \lambda \in \sigma(\delta_A)$ and $e^\lambda = 1$. Thus $\lambda = 2j\pi i$ for some $j \in \mathbb{Z}\setminus\{0\}$. It follows from (2) that there are $\alpha, \beta \in \sigma(A)$ such that $\lambda = \alpha - \beta$. Hence $2|j|\pi = |\lambda| \leq |\alpha| + |\beta| \leq 2r(A) \leq 2\pi$, therefore $|j| \leq 1$. This shows that $\lambda \in [2\pi i, -2\pi i]$.

(d) Since $2\pi i$ and $-2\pi i$ are simple zeros of $f$, there is an entire function $g$ such that

$$f(\lambda) = g(\lambda)(\lambda - 2\pi i)(\lambda + 2\pi i)$$

and $g(\lambda) \neq 0$ for all $\lambda \in \sigma(\delta_A)$, thus $g(\delta_A)$ is invertible on $L(H)$ and

$$f(\delta_A) = g(\delta_A)(\delta_A - 2\pi i)(\delta_A + 2\pi i).$$
Satz 80.3 in [1] gives now
\[ N(f(\delta_A)) = N(\delta_A - 2\pi i) \oplus N(\delta_A + 2\pi i). \]

(e) Take \( C \in N(\delta_A - 2\pi i) \), thus \( CA - AC = 2\pi i C \). Therefore \( 2\pi i C^2 = C^2 A - CAC = C^2 A - (AC + 2\pi i C)C = C^2 A - AC^2 - 2\pi i C^2 \), hence
\[ C^2 A - AC^2 = 4\pi i C^2. \]
Assume that \( C^2 \neq 0 \), then \( 4\pi i \in \sigma(\delta_A) \), therefore, by (2), \( 4\pi \leq 2r(A) \leq 2\pi \), a contradiction. Thus \( C^2 = 0 \). □

Notation. Let \( T \in \mathcal{L}(H) \), \( \rho(T) \subseteq \mathbb{C} \) and \( \rho(T) = \emptyset \). Let \( S(T, \rho(T)) \) be the subset of \( H \) defined by
\[ S(T, \rho(T)) = \bigcap_{\lambda \in \rho(T)} (T - \lambda)(H). \]

Proposition 2. Let \( A \in \mathcal{L}(H) \) be normal.
(a) For \( \mu \in \mathbb{C} \), \( (A - \mu)(H) = (A^* - \overline{\mu})(H) \).
(b) If \( A \) has the spectral resolution (1) and if \( B \in \mathcal{L}(H) \), then
\[ S(A, \rho(B)) = E(\sigma(A) \cap \sigma(B))(H). \]

Proof. (a) Since \( A \) is normal, \( A - \mu \) is normal. Use Exercise 12.36 in [6] to see that \( (A - \mu)(H) = (A^* - \overline{\mu})(H) \).
(b) follows from Theorem 1 in [5]. □

Proposition 3. Let \( A \in \mathcal{L}(H) \) be normal and suppose that \( A \) has the spectral resolution (1). For \( \lambda_0 \in \mathbb{C} \), \( C \in N(\delta_A - \lambda_0) \) and \( D \in N(\delta_A + \lambda_0) \) we have
\[ C(H) \subseteq E(\sigma(A) \cap \sigma(A - \lambda_0))(H) \]
and
\[ D^*(H) \subseteq E(\sigma(A) \cap \sigma(A - \lambda_0))(H). \]

Proof. Take \( C \in \mathcal{L}(H) \) with \( CA - AC = \lambda_0 C \), thus \( AC = C(A - \lambda_0) \). Put \( B = A - \lambda_0 \). For \( \mu \in \rho(B) \) we get
\[ (A - \mu)C(B - \mu)^{-1} = AC(B - \mu)^{-1} - \mu C(B - \mu)^{-1} \]
\[ = CB(B - \mu)^{-1} - \mu C(B - \mu)^{-1} \]
\[ = C(B - \mu)(B - \mu)^{-1} = C. \]
This shows that \( C(H) \subseteq S(A, \rho(B)) \). From Proposition 2(b) we get
\[ S(A, \rho(B)) = E(\sigma(A) \cap \sigma(B))(H), \] (5)
thus \( C(H) \subseteq E(\sigma(A) \cap \sigma(A - \lambda_0))(H) \).
Now take $D \in N(\delta A + \lambda_0)$, hence $DA - AD = -\lambda_0 D$ thus $DA = (A - \lambda_0)D$.

As above let $B = A - \lambda_0$. Then $A^*D^* = D^*B^*$. A similar computation as above gives

$$(A^* - \mu)D^*(B^* - \mu)^{-1} = D^*$$

for all $\mu \in \rho(B^*)$.

Thus $D^*(\mathcal{H}) \subseteq S(A^*, \rho(B^*))$. Since $\rho(B^*) = \{\mu \in \mathbb{C} : \Pi \in \rho(B)\}$, we get from Proposition 2(a) that

$$S(A^*, \rho(B^*)) = S(A, \rho(B)),$$

hence, by (5), $D^*(\mathcal{H}) \subseteq E(\sigma(A) \cap \sigma(A - \lambda_0))$. \quad \Box

**Proposition 4.** Let $A \in \mathcal{L}(\mathcal{H})$ be normal and $r(A) \leq /afii9843i$. 

(a) If $C \in N(\delta A + 2\pi i)$ then $AC = i\pi C$.
(b) If $D \in N(\delta A - 2\pi i)$ then $AD^* = i\pi D^*$ and $DA = i\pi D$.

**Proof.** Suppose that $A$ has the spectral resolution (1). It is clear that

$$\sigma(A) \cap \sigma(A + 2\pi i) \subseteq \{i\pi\}.$$}

From Theorem 12.29 in [6] it follows that $E([i\pi]) = N(A - i\pi)$. Thus

$$E(\sigma(A) \cap \sigma(A + 2\pi i))(\mathcal{H}) \subseteq N(A - i\pi). \quad (6)$$

(a) Take $C \in N(\delta A + 2\pi i)$ and put $\lambda_0 = -2\pi i$. Then $C \in N(\delta A - \lambda_0)$. Proposition 3 and (6) give

$$C(\mathcal{H}) \subseteq N(A - i\pi),$$

hence $AC = i\pi C$.

(b) Take $D \in N(\delta A - 2\pi i)$ and put $\lambda_0 = -2\pi i$. Then $D \in N(\delta A + \lambda_0)$. Proposition 3 and (6) give

$$D^*(\mathcal{H}) \subseteq N(A - i\pi),$$

hence $AD^* = i\pi D^*$. Therefore we have $AD^*x = i\pi D^*x$ for each $x \in \mathcal{H}$. The normality of $A$ gives $A^*D^*x = -i\pi D^*x$ for all $x \in \mathcal{H}$, thus $A^*D^* = -i\pi D^*$, hence $DA = i\pi D$. \quad \Box

We now are in a position to state the main results of this paper. The following theorem is due to Hille [2]. For the convenience of the reader we shall include a proof.

**Theorem 1.** Let $A, B \in \mathcal{L}(\mathcal{H})$ and $e^A = e^B$. If $\sigma(A)$ is incongruent (mod $2\pi i$) then $AB = BA$.

**Proof.** From (4) we get

$$f(\delta_A)(\delta_A(B)) = e^{-A}Be^A - B = 0,$$

thus $AB - BA \in N(f(\delta_A))$. Use Proposition 1(a) and (b) to see that $AB = BA$. \quad \Box
The restriction concerning the spectrum of $A$ in Theorem 1 cannot be dispensed with, as is seen by the following two-dimensional example.

**Example.** Let $A = \pi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $B = \pi \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$.

By induction we see that

\[ A^{2n} = (-1)^n \pi^{2n} I = B^{2n}, \quad A^{2n+1} = (-1)^n \pi^{2n} A, \]

and

\[ B^{2n+1} = (-1)^n \pi^{2n} B \quad \text{for } n = 0, 1, 2, \ldots. \]

This shows that $e^A = -I = e^B$. We have

\[ AB = \pi^2 \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \neq \pi^2 \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = BA. \]

It is easily seen that $A$ is normal and $\sigma(A) = \{i\pi, -i\pi\}$, thus $r(A) = \pi$ and $\sigma(A)$ is not incongruent (mod 2$\pi$). But we have $A^2B = BA^2$, which will be the case in general as the following theorem shows.

**Theorem 2.** Suppose that $A \in \mathcal{L}(\mathcal{H})$ is normal, $B \in \mathcal{L}(\mathcal{H})$ and $e^A = e^B$.

(a) If $r(A) < \pi$ then $AB = BA$.

(b) If $r(A) = \pi$ then $A^2B = BA^2$.

(c) If $r(A) \leq \pi$ and $i\pi \notin \sigma_p(A)$ then $AB = BA$.

(d) If $r(A) \leq \pi$ and $-i\pi \notin \sigma_p(A)$ then $AB = BA$.

**Proof.** As in the proof of Theorem 1 we get $AB - BA \in N(f(\delta_A))$.

(a) If $r(A) < \pi$, then $\sigma(A)$ is incongruent (mod $2\pi$), hence $AB = BA$, by Theorem 1.

(b) From Proposition 1 we see that there are operators $C \in N(\delta_A + 2\pi i)$ and $D \in N(\delta_A - 2\pi i)$ such that $AB - BA = C + D$. From $CA - AC = -2\pi i C$ and Proposition 4(a) we derive $AC = i\pi C$, hence

\[ CA = AC - 2\pi i C = -i\pi C = -AC, \]

thus

\[ AC + CA = 0. \quad (7) \]

Use Proposition 4(b) and $DA - AD = 2\pi iD$ to get $DA = i\pi D$ and

\[ AD = DA - 2\pi i D = -i\pi D = -DA, \]
hence
\[ AD + DA = 0. \]  
(8)

From \( AB - BA = C + D \) it follows that
\[ A^2B - ABA = AC + AD \]  
(9)

and
\[ ABA - B^2 = CA + DA. \]  
(10)

The addition of (9) and (10) shows
\[ A^2B - BA^2 = AC + CA + AD + DA. \]

Now use (7) and (8) to get
\[ A^2B = BA^2. \]  
(c) As in (b) we have \( AB - BA = C + D \) with \( C \in N(\delta A + 2\pi i) \) and \( D \in N(\delta A - 2\pi i) \). Since \( N(A - i\pi) = \{0\} \) we conclude from Proposition 4 that \( C = 0 = D^* \), thus \( C = D = 0 \), hence \( AB = BA \).

(d) If \(-i\pi \notin \sigma_p(A)\) then \( i\pi \notin \sigma_p(-A)\). From \( e^{-A} = e^{-B} \) we get then \( AB = BA \) as in the proof of (c). □

Now suppose that the spectrum \( \sigma(A) \) of \( A \in \mathcal{L}(\mathcal{H}) \) satisfies
\[ \sigma(A) \subseteq \{ z \in \mathbb{C} : |\text{Im} \ z| \leq \pi \} \]  
(11)

and
\[ \sigma(A) \cap \sigma(A + 2\pi i) \subseteq \{ i\pi \}. \]  
(12)

Then it is easily seen that \( M_A \subseteq \{2\pi i, -2\pi i\} \) and
\[ N(f(\delta A)) = N(\delta A - 2\pi i) \oplus N(\delta A + 2\pi i). \]

Let \( A \) be normal and suppose that (11) and (12) hold. Then it is easy to see that the statements of Proposition 4 remain valid. Thus an inspection of the proof of Theorem 2 shows the following result:

**Theorem 3.** Suppose that \( A \in \mathcal{L}(\mathcal{H}) \) is normal, \( B \in \mathcal{L}(\mathcal{H}) \), \( e^A = e^B \) and \( \sigma(A) \) satisfies (11) and (12). Then \( A^2B = BA^2 \). If \( i\pi \notin \sigma_p(A) \) or \(-i\pi \notin \sigma_p(A)\) then \( AB = BA \).

**Theorem 4.** Let \( A, B \in \mathcal{L}(\mathcal{H}) \), let \( e^A = e^B \) and let \( \sigma(A) \) and \( \sigma(A - B) \) be incongruent (mod \( 2\pi i \)). Then there is some \( k \in \mathbb{Z} \) with
\[ A - B = (2k\pi i)I. \]

**Proof.** From Theorem 1 we get \( AB = BA \), thus \( e^{A-B} = I \). Put \( C = A - B \) and let \( g(z) = e^z - 1 \) (\( z \in \mathbb{C} \)). Hence \( g(C) = 0 \). Take \( \lambda, \mu \in \sigma(C) \), then \( e^\lambda = e^\mu = 1 \), thus \( \lambda - \mu = 2j\pi i \) for some \( j \in \mathbb{Z} \). Since \( \sigma(C) \) is incongruent (mod \( 2\pi i \),...
we get \( \lambda = \mu \). This shows that there is \( k \in \mathbb{Z} \) such that \( \sigma(C) = \{2k\pi i\} \). Since \( 2k\pi i \) is a simple zero of \( g \), there is an entire function \( h \) with \( g(\lambda) = h(\lambda)(\lambda - 2k\pi i) \) and \( h(2k\pi i) \neq 0 \).

This gives

\[
0 = g(C) = h(C)(C - 2k\pi i).
\]

Since \( h(2k\pi i) \neq 0 \), there is an entire function \( \sigma(C) = \{\frac{2k}{a}i\} \). Since \( \sigma(C) \) is a simple zero of \( g \), there is an entire function \( h \) with \( g(\lambda) = h(\lambda)(\lambda - \frac{2k}{a}i) \) and \( h(\frac{2k}{a}i) = 0 \).

This gives

\[
0 = g(C) = h(C)(C - \frac{2k}{a}i).
\]

Since \( h(C) \) is invertible, \( C = \frac{2k}{a}iI \). \( \square \)

As an immediate consequence of Theorem 4 we have the following well-known result:

**Corollary 1.** If \( A, B \in L(H) \) are selfadjoint and if \( e^A = e^B \) then \( A = B \).

**Proof.** From \( \sigma(A), \sigma(A - B) \subseteq \mathbb{R} \) we see that \( \sigma(A) \) and \( \sigma(A - B) \) are incongruent (mod \( 2\pi i \)). Theorem 4 gives \( A - B = 2k\pi iI \) for some \( k \in \mathbb{Z} \). Since \( A - B = (A - B)^* = -2k\pi iI = B - A, A = B \). \( \square \)

**Corollary 2.** If \( A, B \in L(H) \) are normal and if \( e^A = e^B \) then \( A + A^* = B + B^* \).

**Proof.** Since \( A \) and \( B \) are normal, we see that

\[
e^{A + A^*} e^{A^*} = e^A (e^A)^* = e^B (e^B)^* = e^B e^{B^*} = e^{B + B^*}.
\]

We now use Corollary 1. \( \square \)

For our next result we need the following proposition.

**Proposition 5.** Suppose that \( A \) and \( B \) are normal operators in \( L(H) \), \( r(A) \leq \pi, r(B) \leq \pi \) and \( AB = BA \). Then

(a) \( A - B \) is normal;

(b) \( N(A - B - 2\pi i) = N(A - i\pi) \cap N(B + i\pi) \).

**Proof.** (a) From \( AB = BA \) we get \( AB^* = B^*A \) and \( A^*B = BA^* \) by the Fuglede–Putnam–Rosenblum Theorem [6, Theorem 12.16]. A simple computation gives then that \((A - B)(A - B)^* = (A - B)^*(A - B)\).

(b) It is clear that \( N(A - i\pi) \cap N(B + i\pi) \subseteq N(A - B - 2\pi i) \). Put \( C = A - B \) and take \( x_0 \in N(A - B - 2\pi i) \). We can assume that \( \|x_0\| = 1 \). For the following computations let \( \langle \cdot | \cdot \rangle \) denote the inner product \( H \). From \( Cx_0 = 2\pi ix_0 \) and (a) we get

\[
A^*x_0 = B^*x_0 - 2\pi ix_0 \quad \text{and} \quad A^*x_0 = B^*x_0 - 2\pi ix_0.
\]

(13)
Put \( D = i(B - B^*) \). Then \( D^* = D \) and
\[
(Dx_0|x_0) = i((Bx_0|x_0) - (B^*x_0|x_0))
\]
\[
= -2 \text{Im}(Bx_0|x_0).
\]
From \(|\text{Im}(Bx_0|x_0)| \leq |(Bx_0|x_0)| \leq \|B\| = r(B) \leq \pi\), we see that \(-\pi \leq \text{Im}(Bx_0|x_0)\). Thus
\[
(Dx_0|x_0) \leq 2\pi. \quad (14)
\]
Now use (13) and (14) to derive
\[
\|((A - i\pi)x_0)^2 = ((A^* + i\pi)(A - i\pi)x_0|x_0)
\]
\[
= \|A^*x_0\|^2 + i\pi((A - A^*)x_0|x_0) + \pi^2
\]
\[
\leq 2\pi^2 + \pi((A - A^*)x_0|x_0)
\]
\[
= 2\pi^2 + i\pi((B - B^* + 4\pi)\text{Im}(Bx_0))
\]
\[
= -2\pi^2 + \pi(Dx_0|x_0)
\]
\[
\leq -2\pi^2 + 2\pi^2 = 0.
\]
Thus \( x_0 \in N(A - i\pi) \), hence, by (13), \( x_0 \in N(B + i\pi) \). \(\square\)

**Theorem 5.** Suppose that \( A \) and \( B \) are normal operators in \( \mathcal{L}(H) \), \( r(A) \leq \pi \), \( r(B) \leq \pi \), and \( e^A = e^B \).

(a) If \( i\pi \notin \sigma_p(A) \) then \( AB = BA \) and \( -(1/2\pi i)(A - B) \) is an orthogonal projection.

(b) If \( -i\pi \notin \sigma_p(A) \) then \( AB = BA \) and \( (1/2\pi i)(A - B) \) is an orthogonal projection.

(c) If \( -i\pi \notin \sigma_p(A) \) and \( i\pi \notin \sigma_p(A) \) then \( A = B \).

(d) If \( -i\pi \notin \sigma_p(A) \) and \( -i\pi \notin \sigma_p(B) \) then \( A = B \).

(e) If \( i\pi \notin \sigma_p(A) \) and \( i\pi \notin \sigma_p(B) \) then \( A = B \).

**Proof.** Let \( C = A - B \).

(a) Theorem 2 gives \( AB = BA \), hence \( e^C = I \). Use Proposition 5 to see that \( C \) is normal. Define the polynomial \( p \) by \( p(z) = z(z + 2\pi i)(z - 2\pi i) \). Then \( p(C) \) is normal and therefore
\[
\|p(C)\| = r(p(C)) \quad (15)
\]

Now take \( \lambda \in \sigma(C) \). Since \( e^C = I \), \( \lambda = 2j\pi i \) for some \( j \in \mathbb{Z} \). Thus
\[
2|j|\pi = |\lambda| \leq \|C\| \leq \|A\| + \|B\| = r(A) + r(B) \leq 2\pi,
\]
hence \( j \in \{0, \pm1\} \), thus \( \sigma(C) \subseteq \{0, 2\pi i, -2\pi i\} \). The spectral mapping theorem gives now
\[
\sigma(p(C)) = p(\sigma(C)) = \{0\}.
\]
hence, by (15),
\[(C - 2\pi i)(C + 2\pi i)C = 0.\] (16)
Since \(N(A - i\pi) = \{0\}\), we see from Proposition 5(b) and (16) that \(C^2 = -2\pi iC\). Put \(P = -(1/2\pi i)C\). Then \(P^2 = P\), thus \(P\) is a projection. It remains to show that \(P = P^*\). But this follows from Corollary 2, since
\[
P^* = \frac{1}{2\pi i}(A^* - B^*) = -\frac{1}{2\pi i}(A - B) = P.
\]
(b) If \(-\pi \notin \sigma_p(A)\) then \(\pi \notin \sigma_p(-A)\). Since \(e^{-A} = e^{-B}\), (a) shows that \(-1/2\pi i\)
\((-A - (-B)) = (1/2\pi i)(A - B)\) is an orthogonal projection.
(c) From (a) and (b) we conclude that \(-2\pi iC = C^2 = 2\pi iC\), hence \(C = 0\).
(d) From (b) we get \(C^2 = 2\pi iC\). Use (b) with \(B\) instead of \(A\) to derive \(C^2 = (B - A)^2 = (1/2\pi i)(B - A) = -2\pi iC\). Hence \(C = 0\).
(e) is now clear.

Remark. Theorem 5 generalizes Satz 4 in [7].

For our final result in this paper we return to the situation of Theorem 3.

**Theorem 6.** Suppose that \(A \in \mathcal{L}(H)\) is normal, \(B \in \mathcal{L}(H)\), \(e^A = e^B\) and \(\sigma(A)\) satisfies \((11)\) and \((12)\). Then we have:

(a) \(A^{2n}B = BA^{2n}\) for all \(n \in \mathbb{N}\).

(b) \(A^{2n+1}B - BA^{2n+1} = A^{2n}(AB - BA) = (-1)^n \pi^{2n}(AB - BA)\) for all \(n \in \mathbb{N}\).

(c) \(e^{2B}(AB - BA) = e^{2B}(AB - BA) = AB - BA\).

(d) If \(AB \neq BA\) then there is some \(k \in \mathbb{Z}\) such that \(k\pi \in \sigma_p(B)\).

**Proof.** (a) Follows from Theorem 3.

(b) As in the proof of Theorem 2 there are operators \(C \in N(\delta_A + 2\pi i)\) and \(D \in N(\delta_A - 2\pi i)\) such that \(AB - BA = C + D\), \(AC = i\pi C\) and \(DA = i\pi D\). It follows that
\[
i\pi(AB - BA) = i\pi C + i\pi D = AC + DA
= AC + AD + 2\pi iD
= A(AB - BA) + 2\pi iD,
\]
thus
\[
(A - i\pi)(AB - BA) = -2\pi iD.
\]
Since \(D(A - i\pi) = 0\),
\[
(A - i\pi)(AB - BA)(A - i\pi) = 0.
\]
This and
\[
A(AB - BA) = A^2B - ABA = BA^2 - ABA
= (BA - AB)A = -(AB - BA)A
\]
show that
\[ 0 = (A - i\pi)((-A - i\pi)(AB - BA)) = -(A^2 + \pi^2)(AB - BA), \]
hence
\[ A^2(AB - BA) = -\pi^2(AB - BA). \tag{17} \]

Take \( n \in \mathbb{N} \). Then, by (a) and (17),
\[ A^{2n+1}B - BA^{2n+1} = A^{2n+1}B - A^{2n}BA = A^{2n}(AB - BA) \]
\[ = (-1)^n \pi^{2n}(AB - BA). \]

(c) Since (b) holds we get
\[ \frac{1}{2}(e^A - e^{-A})(AB - BA) = \sinh(A)(AB - BA) \]
\[ = \sum_{n=0}^{\infty} \frac{A^{2n+1}(AB - BA)}{(2n+1)!} \]
\[ = A \sum_{n=0}^{\infty} \frac{A^{2n}(AB - BA)}{(2n+1)!} \]
\[ = A \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}(AB - BA)}{(2n+1)!} \]
\[ = \frac{A}{\pi} \left( \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} \right)(AB - BA) \]
\[ = \frac{\sin \pi}{\pi} A(AB - BA) = 0. \]

Thus \( e^A(AB - BA) = e^{-A}(AB - BA) \). From \( e^A = e^B \) we then derive
\[ e^{2B}(AB - BA) = e^{2A}(AB - BA) = AB - BA. \]

(d) Suppose that \( AB \neq BA \). From (c) we see that \( 1 \in \sigma_p(e^{2B}) \). The spectral mapping theorem for the point spectrum ([6, Theorem 10.33]) shows that there is \( \lambda \in \sigma_p(B) \) such that \( e^{2\lambda} = 1 \), hence \( 2\lambda = 2k\pi i \) for some \( k \in \mathbb{Z} \). This gives \( k\pi i \in \sigma_p(B) \). \( \square \)

**Corollary 3.** Suppose that \( A \in \mathcal{L}(\mathcal{H}) \) is normal, \( B \in \mathcal{L}(\mathcal{H}) \), \( e^A = e^B \), \( \sigma(A) \) satisfies (11) and (12) and \( \sigma(B) \) satisfies
\[ \sigma_p(B) \cap \{k\pi i : k \in \mathbb{Z}\} = \emptyset \]
then \( AB = BA. \)
References