On elliptic non-divergence operators with measurable coefficients

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Abstract

We study properties of the coefficient matrices of non-divergence operators on $\mathbb{R}^n$ aiming at sectoriality and $R$-sectoriality of these operators. In particular, we present results on approximation, scaling, and the behaviour in the $L_p$-scale.

1 Introduction

Let $m \in \mathbb{N}$. In this paper we study sectoriality and $R$-sectoriality of elliptic operators of order $2m$ in $L_p(\mathbb{R}^n, \mathcal{A})$, $p \in (1, \infty)$, which have the following form

$$A = \sum_{|\alpha|=2m} a_\alpha(x)D^\alpha, \quad D(A) = W^{2m}_p(\mathbb{R}^n, \mathcal{A}),$$

where $a_\alpha : \mathbb{R}^n \to \mathcal{A}$ are bounded measurable functions. Here we use the usual multi-index notation $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and $D^\alpha = D_{\alpha_1} \ldots D_{\alpha_n}$ where $D_j := -i\partial/\partial x_j$ for $j = 1, \ldots, n$.

The classical three step approach to show sectoriality of non-divergence operators can be described as follows: First study operators with constant coefficients. Second study small perturbations of constant coefficient operators. In the last step use localization and patch together the localized operators (we refer in particular to the localization scheme in [1] for this last step). The approach has also been successfully used to establish a number of further properties such as maximal $L_p$-regularity, $R$-sectoriality, boundedness of imaginary powers (BIP), or boundedness of an $H^\infty$-functional calculus. For the second and third step, it seems to be unavoidable to assume some kind of smoothness for the highest order coefficients, e.g., uniform continuity ([1], [9], [13], [14], [5]) or VMO ([10]) for the properties of sectoriality and $R$-sectoriality, and Hölder continuity for BIP ([16]) or a bounded $H^\infty$-calculus ([1], [6]) (at least, Hölder continuity is needed for boundary value problems; on
\( \mathbb{R}^n \), however, it is known that uniform continuity ([7]) or even VMO ([8]) is sufficient. It is also very well known that for all these properties lower order terms \( \sum_{|\beta|<2m} a_\beta(x) D^\beta \) with bounded and measurable \( a_\beta \) may be studied by perturbation methods. Hence we may neglect lower order terms in our study.

In this paper we somehow change the point of view. We make no assumptions on the smoothness of the coefficients and we do not use localization. Instead, we investigate what can be said in the general case by looking not on a particular operator in a particular \( L_p \), but on sets of operators in a particular \( L_p \) or on a particular operator in a scale of \( L_p \)-spaces. On the technical side we limit ourselves to abstract nonsense and hand-waving. Consequently, our observations are not of an absolute (certain regularities imply certain properties) but rather of a relative nature. Nevertheless, they give some useful information.

The paper is organized as follows: in Section 2 we study approximation of coefficients and prove a result (Theorem 1) on approximation in the strong operator topology (not in uniform operator topology, the latter being, of course, very well known and usually used for bounded uniformly continuous coefficients). This result on sectoriality also holds for \( R \)-sectoriality and makes clear how a precise control on the constants may be exploited. The result on \( R \)-sectoriality is new, and even the result on sectoriality has, to our knowledge, not been explicitly stated before. We recall that, for operators in divergence form, scaling may sometimes be used to improve constants in certain estimates (cf. [2]). In Section 3 we concentrate on non-divergence operators with coefficients that are invariant under scaling. On the one hand, this additional assumption makes life more easy and, in fact, the results are somehow analogous to those for operators in divergence form. On the other hand, however, famous counterexamples in elliptic theory belong to this class. In Section 4, the last one, we use a result from abstract interpolation theory to investigate the behaviour of a given operator in a scale of \( L_p \)-spaces. These results are also new.

**Notation:** We denote tuples \( (a_\alpha)_{|\alpha|\leq 2m} \) of functions \( a_\alpha \in L_\infty(\mathbb{R}^n, \mathcal{F}) \) by \( a \) and the set of all such \( a \) by \( \mathcal{A}^{2m} \). For a given tuple of coefficients \( a \in \mathcal{A}^{2m} \) and \( p \in (1, \infty) \) fixed we define the associated operator in \( L_p(\mathbb{R}^n, \mathcal{F}) \) by

\[
(\text{Op}(a)u)(x) := \sum_{|\alpha|=2m} a_\alpha(x)(D^\alpha u)(x), \quad x \in \mathbb{R}^n.
\]

where \( u \in D(\text{Op}(a)) := W_p^{2m}(\mathbb{R}^n, \mathcal{F}) \). We define

\[
\mathcal{A}_p^{2m} := \{ a \in \mathcal{A}^{2m} : \text{Op}(a) \text{ is quasi-sectorial in } L_p(\mathbb{R}^n, \mathcal{F}) \}.
\]

Recall that an operator \( A \) in a Banach space \( X \) is called sectorial if, for some \( \theta_0 \in [0, \pi) \), the spectrum \( \sigma(A) \) of \( A \) is contained in the closure of the open sector \( \Sigma_{\theta_0} := \{ z \in \mathbb{C} \setminus \{ 0 \} : |\arg z| \leq \theta_0 \} \) and, for all \( \theta \in (0, \pi - \theta_0) \), the set \( \{ \lambda(\lambda + A)^{-1} : \lambda \in \Sigma_\theta \} \) is bounded. The operator \( A \) is called quasi-sectorial if \( \nu + A \) is sectorial for some \( \nu \geq 0 \). If, in these definitions, the set \( \{ \lambda(\lambda + A)^{-1} : \lambda \in \Sigma_\theta \} \) is even \( R \)-bounded, then the operator \( A \) is called \( R \)-sectorial or quasi-\( R \)-sectorial, respectively. We denote the set of coefficients \( a \) such that \( \text{Op}(a) \) is quasi-\( R \)-sectorial in \( L_p(\mathbb{R}^n, \mathcal{F}) \) by \( \mathcal{R}_p \mathcal{A}_p^{2m} \). Recall that a set \( \mathcal{T} \) of linear operators
$X \to Y$ is called $R$-bounded, if there is a constant $C > 0$ such that, for all $k \in \mathbb{N}$, $T_1, \ldots, T_k \in \mathbb{T}$, and $x_1, \ldots, x_k \in X$, one has

$$\|E\| \sum_{j=1}^{k} \varepsilon_j T_j x_j \|_Y \leq C \|E\| \sum_{j=1}^{k} \varepsilon_j x_j \|_X,$$

where $(\varepsilon_j)$ denotes a sequence of independent and symmetric, $\{\pm 1\}$-valued random variables on a probability space, e.g. the Rademachers. The infimum of all such $C$ is called the $R$-bound of $T$. For the notions of $R$-boundedness, $R$-sectoriality, and their connection with maximal $L_p^\tau$-regularity we refer to [14].

Since all functions are complex-valued on $\mathbb{R}^n$ we drop $\mathbb{R}^n$ and $\mathcal{T}$ in notation and write only $L_p$ and $W_p^{2m}$.

## 2 Sets of coefficients and approximation

On $\mathcal{M}^{2m}$ we consider the norm $\|a\|_\infty := \sum_{|\alpha| = m} \|a_\alpha\|_\infty$. For any $q \in [1, \infty]$ we denote by $\tau_q$ the topology induced by the strong operator topology when considering the elements $a_\alpha$ of $\mathcal{A}$ as pointwise multipliers on $L^q(\mathbb{R}^n)$. Then $\tau_1 \subset \tau_q \subset \tau_\infty$ for $1 \leq q \leq r \leq \infty$ and $\tau_\infty$ is induced by $\| \cdot \|_\infty$. Note that $a_k \to a (k \to \infty)$ in $\tau_q$ for $q \in [1, \infty)$ if $(a_k)$ is a sequence such that $\sup_k \|a\|_\infty < \infty$, and $a_k \to a (k \to \infty)$ in point- and componentwise a.e.

In the following $p \in (1, \infty)$ is fixed. For a tuple $a$ of coefficients and $A = \text{Op}(a)$ we introduce the following notation for the constants appearing in a priori estimates, where $\nu \geq 0$ and $\theta \in (0, \pi)$:

$$P_{\theta, \nu}(a) := \inf \{ M > 0 : \forall \lambda \in \Sigma_\theta, u \in W_p^{2m} : \|\lambda u\|_p + \|(-\Delta)^m u\|_p \leq M \|(\lambda + \nu + A) u\|_p \}.$$ 

We use the convention $\inf \emptyset := \infty$. For $\nu = 0$ we just write $P_{0}(a) = P_{\theta, 0}(a)$. We also shall use the constants in resolvent estimates:

$$M_{\theta, \nu}(a) := \sup \{ \|\lambda(\lambda + \nu + A)^{-1}\|_{p \to p} + \|(-\Delta)^m(\lambda + \nu + A)^{-1}\|_{p \to p} : \lambda \in \Sigma_\theta \}.$$ 

Again we let $M_{\theta} := M_{\theta, 0}(a)$ and understand $M_{\theta, \nu}(a) = \infty$ if $-\Sigma_\theta \not\subset \rho(\nu + A)$. Since we are also interested in $R$-sectoriality, we introduce the corresponding $R$-boundedness counterparts. We denote by $P_{\theta, \nu}^R(a)$ the infimum of all constants $C > 0$ such that, for all $k \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_k \in \Sigma_\theta$, and $u_1, \ldots, u_k \in W_p^{2m}$ we have

$$\|E\| \sum_{j=1}^{k} \varepsilon_j \lambda_j u_j \|_p + \|E\| \sum_{j=1}^{k} \varepsilon_j (-\Delta)^m u_j \|_p \leq C \|E\| \sum_{j=1}^{k} \varepsilon_j (\lambda_j + \nu + A) u_j \|_p,$$

and we denote by $R_{\theta, \nu}(a)$ the $R$-bound of the set

$$\{ (\lambda(\lambda + \nu + A)^{-1}, (-\Delta)^m(\lambda + \nu + A)^{-1}) : \lambda \in \Sigma_\theta \}.$$
considered as operators from \( L_p \) into \( L_p \times L_p \). Recall that \( p \) was fixed. If we want to indicate the special \( L^p \)-space to which we refer then we use \( p \) as an additional subscript, e.g., \( M_{\theta,\nu,p}(a) \) etc.

We have the following theorem on approximation of coefficients.

**Theorem 1.** Let \( p \in (1, \infty) \) and \( \mathcal{H} \subset \mathcal{S}^{2m}_p \) be convex and \( \| \cdot \|_{\infty} \)-bounded. Suppose that \( M, \nu \geq 0 \) and \( \theta \in (0, \pi) \) are such that \( M_{\theta,\nu} \leq M \) on \( \mathcal{H} \). Then \( \mathcal{H}_{\tau_p} \subset \mathcal{S}^{2m}_p \) and \( M_{\theta,\nu} \leq M \) on \( \mathcal{H}_{\tau_p} \).

If, in addition, \( \mathcal{K} \subset \mathcal{R} \mathcal{S}^{2m}_p \) and \( R_{\theta,\nu} \leq M \) on \( \mathcal{K} \), then \( \mathcal{K}_{\tau_p} \subset \mathcal{R} \mathcal{S}^{2m}_p \) and \( R_{\theta,\nu} \leq M \) on \( \mathcal{K}_{\tau_p} \).

**Proof.** First we notice that \( M_{\theta,\nu} \leq M \) on \( \mathcal{H} \) implies \( P_{\theta,\nu} \leq M \) on \( \mathcal{H} \). Then we observe that \( P_{\theta,\nu} \) is lower semi-continuous for \( \tau_p \) on \( \mathcal{S}^{2m} \) which is immediate from the definitions. Hence \( P_{\theta,\nu} \leq M \) on \( \mathcal{H}_{\tau_p} \). For \( a \in \mathcal{H}_{\tau_p} \) we choose \( a_0 \in \mathcal{K} \) and consider \( a_\mu := (1-\mu)a_0 + \mu a \) for \( \mu \in [0,1] \). Since \( \mathcal{H}_{\tau_p} \) is again convex, we have \( P_{\theta,\nu}(a_\mu) \leq M \) for all \( \mu \in [0,1] \). Since \( a_0 \in \mathcal{K} \subset \mathcal{S}^{2m}_p \), the continuity method yields the first assertion.

Since \( \mathcal{K}_{\tau_p} \subset \mathcal{S}^{2m}_p \), the assertion \( R_{\theta,\nu} \leq M \) on \( \mathcal{K}_{\tau_p} \) is equivalent to \( P_{\theta,\nu}^R \leq M \) on \( \mathcal{K}_{\tau_p} \). Now we notice that

\[
\|E\| \sum_{j=1}^k \varepsilon_j f_j p = \frac{1}{2^k} \sum_{\sigma_j=\pm 1} \| \sum_{j=1}^k \sigma_j f_j \|_p, \tag{3}
\]

and the RHS hereof is just an average over choices of signs in the sum. Hence, if we let \( a_m \to a \) in \( \tau_p \), then (3) shows that (2) for \( \text{Op}(a_m) \) in place of \( \text{Op}(a) \) in place of \( A \) will converge to (2) for \( \text{Op}(a) \) in place of \( A \). Thus \( P_{\theta,\nu}^R \leq M \) on \( \mathcal{H} \) implies \( P_{\theta,\nu}^R \leq M \) on \( \mathcal{K}_{\tau_p} \), which in turn implies \( R_{\theta,\nu} \leq M \) on \( \mathcal{K}_{\tau_p} \). \( \square \)

**Remark 2.** The proof shows that the convexity assumption may be weakened. It is sufficient that \( \mathcal{H}_{\tau_p} \) is arc-wise connected for \( \tau_{\infty} \).

For applicability of the above results we want to mention the following.

**Remark 3.** For a tuple \( a = (a_\alpha)_{|\alpha|=2m} \in \mathcal{S}^{2m} \) we denote the associated symbol by

\[
\text{Sy}(a, x, \xi) := \sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha, \quad x, \xi \in \mathbb{R}^n,
\]

where, as usual, \( \xi^\alpha := \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \) for \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n \). If \( K \subset \{ z \in \mathcal{G} : \text{Re} \ z > 0 \} \) is compact and convex then

\[
\mathcal{S}^{2m}_K := \{ a \in \mathcal{S}^{2m} : \text{ for a.e. } x \in \mathbb{R}^n, \forall |\xi| = 1 : \text{Sy}(a, x, \xi) \in K \}
\]

is convex. Let \( a = (a_\alpha) \in \mathcal{S}^{2m}_K \). Taking a standard mollifier sequence \( (\rho_k) = (k^n \rho(k)) \) we have that all \( a_k := (\rho_k * a_\alpha)_{|\alpha|=2m} \), \( k \in \mathbb{N} \), belong to \( \mathcal{S}^{2m}_K \), and that \( a_k \to a \) \( (k \to \infty) \) in \( \tau_q \) for any \( q \in [1, \infty] \). Hence via Theorem 1 a good control on sectoriality \([R\text{-sectoriality}]\) constants for smooth coefficients yields sectoriality \([R\text{-sectoriality}]\) also for non-smooth coefficients.
3 Scaling and scaling invariant coefficients

We define, for \( \mu > 0 \), the dilation operator \( S_\mu f := f(\mu \cdot) \) and extend it to tuples \( a = (a_\alpha)_{|\alpha|=m} \in \mathcal{S}^{2m} \) via \( S_\mu a = (S_\mu a_\alpha)_{|\alpha|=m} \). Observe that \( \|S_\mu f\|_p = \mu^{-n/p}\|f\|_p \) for \( f \in L_p(\mathbb{R}^n) \) and \( p \in (1, \infty) \). For \( \mu > 0 \) and \( a \in \mathcal{S}^{2m} \) we have
\[
S_{1/\mu} \text{Op}(a) S_\mu u = \mu^{2m} \text{Op}(S_{1/\mu} a) u
\] (4)
for \( u \in W_p^{2m} \). Since \( S_\mu \) is an isomorphism of \( L_p \) with inverse \( S_{1/\mu} \), we conclude that \( S_\mu \) leaves also \( \mathcal{S}_p^{2m} \) invariant.

**Lemma 4.** Let \( p \in (1, \infty) \), \( a \in \mathcal{S}^{2m} \) and \( \theta \in (0, \pi) \), \( \nu \geq 0 \). For any \( \mu > 0 \) we have
\[
K_{\theta,\nu}(a) = K_{\theta,\nu_{1/\mu}}(S_{1/\mu} a) \quad \text{where } K \in \{P, M, R, P^R\}.
\] (5)

**Proof.** The other cases being similar we only study \( K = P \). In
\[
\|\lambda u\|_p + \|(-\Delta)^m u\|_p \leq M \| (\lambda + \nu + \text{Op}(a)) u \|_p
\]
we write \( u = S_\mu v \) and \( |S_{1/\mu}| \|_p \) in place of \( \| \cdot \|_p \). By (4) we obtain
\[
\|\lambda u\|_p + \|\mu^{2m}(-\Delta)^m u\|_p \leq M \| (\lambda + \nu + \mu^{2m} \text{Op}(S_{1/\mu} a)) u \|_p.
\]
Dividing by \( \mu^{2m} \) we obtain the assertion since \( \mu^{2m} \Sigma_\theta = \Sigma_\theta \).

**Remark 5.** If we have \( \nu = 0 \) in the preceding lemma we see that the constants cannot depend on, e.g., the modulus of continuity or the Hölder norm of the \( a_\alpha \) in \( a \). This makes finiteness of the constants for \( \nu = 0 \) the most interesting case.

We call an \( a \in \mathcal{S}^{2m} \) scaling-invariant if \( S_\mu a = a \) for all \( \mu > 0 \). For scaling-invariant coefficients we obtain the following.

**Proposition 6.** Let \( p \in (1, \infty) \) and \( a \in \mathcal{S}^{2m} \) be scaling-invariant. For any \( \theta \in (0, \pi) \) and \( \nu > 0 \) we have
\[
K_{\theta,\nu}(a) = K_{\theta}(a) \quad \text{where } K \in \{P, M, R, P^R\}.
\] (6)
In particular, \( P_{\theta,\nu}(a) < \infty \) for some \( \theta, \nu \) implies the Gårding inequality
\[
\| \text{Op}(a) u \|_p \geq (P_{\theta}(a))^{-1} \|u\|_p, \quad u \in W_p^{2m}.
\] (7)
Moreover, if \( (0, \infty) \cap \rho(-\text{Op}(a)) \neq \emptyset \) then \( \text{Op}(a) \in \mathcal{S}_p^{2m} \).

**Proof.** For (6) we use (5) in Lemma 4 and let \( \mu \to \infty \) (where we may use Theorem 1). We obtain (7) by letting \( \lambda \to 0 \) in the a priori estimate. To prove the last assertion let \( \lambda_0 > 0 \) be in \( \rho(-\text{Op}(a)) \) and \( M := \| (\lambda_0 + \text{Op}(a))^{-1} \|_{p-p} \). By (4) and scaling-invariance we have, for any \( \mu > 0 \) and \( A = \text{Op}(a) \),
\[
M = \| S_{1/\mu}(\lambda_0 + A)^{-1} S_\mu \|_{p-p} = \| (\lambda_0 + \mu^{2m} A)^{-1} \|_{p-p} = \| \mu^{-2m} (\lambda_0 \mu^{-2m} + A)^{-1} \|_{p-p},
\]
which means \( \|\lambda(\lambda + A)^{-1}\|_{p-p} = M\lambda_0 \) for any \( \lambda > 0 \).
For general \( a \in \mathcal{S}^{2m} \) we would obtain (7) only for \( \nu + \text{Op}(a) \) and some \( \nu > 0 \) in place of \( \text{Op}(a) \).

Examples of scaling-invariant operators are, of course, constant coefficient operators, for which the technique is well-known. Other examples are given by operators

\[-(\Delta + c \sum_{j,k=1}^{n} \frac{x_j x_k}{|x|^2} \partial_j \partial_k), \quad c \geq 0.\]

For \( n \geq 3 \), operators of this type provide well-known counterexamples in the theory of elliptic boundary value problems (we refer to [15, Sect. 1.1]). It seems, however, not to be clear, if they also serve as counterexamples to sectoriality on \( \mathbb{R}^n \).

4 Extrapolation via interpolation

In this section we exploit Sneiberg’s Lemma (cf. [2, Lem. 23, p. 53], cf. also [12]):

**Lemma 7 (Sneiberg’s Lemma).** Let \((X_\theta)\) and \((Y_\theta)\) complex interpolation scales of reflexive spaces where \( \theta \in [0,1] \). If \( S \) is a linear operator which is bounded \( X_\theta \rightarrow Y_\theta \) for each \( \theta \in [0,1] \). Then the following subsets of \((0,1)\) are open:

\[\{ \theta : \exists \eta > 0 : \|Sx\|_{Y_\theta} \geq \eta \|x\|_{X_\theta} \} , \quad \{ \theta : S : X_\theta \rightarrow Y_\theta \text{ is an isomorphism} \} .\]

The second statement follows from the first if we also consider the dual operator \( S^* \) in the dual scales \((Y_\theta^*)\) and \((X_\theta^*)\).

**Theorem 8.** For fixed \( a \in \mathcal{S}^{2m} \), \( \theta \in (0,\pi) \), the following subsets of \((1,\infty)\) are open:

\[\{ p \in (1,\infty) : K_{\theta,0,p} < \infty \} \quad \text{where } K \in \{ P, M, R, P^R \}.\]

For \( \nu > 0 \) also the subsets

\[\{ p \in (1,\infty) : K_{\theta,\nu,p} < \infty \text{ and } \nu + \text{Op} \text{ is invertible in } L_p \} \quad \text{where } K \in \{ P, M, R, P^R \}\]

are open.

**Proof.** We choose a dense sequence \((\lambda_j)\) in \( \Sigma_\theta \) and let \( A := \text{Op}(a) \). We first treat the case \( \nu = 0 \) and start with \( K \in \{ P, M \} \). We shall apply Lemma 7 to

\[X_p := \{ (u_j) \in (W^2_{p^{2m}})^N : \| (u_j) \|_{X_p} := (\sum_j (\| \lambda_j u_j \|_p + \| (-\Delta)^{m} u_j \|_p)^{p/2})^{1/p} < \infty \}\]

and \( Y_p := L_p(L_p) \) and the operator \( S : (u_j) \mapsto ((\lambda_j + A)u_j) \). Observe that

\[\| (\lambda_j + A)u_j \|_p \leq \| \lambda_j u_j \|_p + \| Au_j \|_p \leq \| \lambda_j u_j \|_p + C \| (-\Delta)^{m} u_j \|_p\]
where $C$ depends on $p$ and $\|a\|_\infty$. This yields boundedness of $S : X_p \to Y_p$ for any $p \in (1, \infty)$. Now Lemma 7 proves the claim.

For $K \in \{P^R, R\}$ we use the same operator $S$ but we use

$$X_p := \{(u_j) \in (W^{2m}_p)^N : \| (u_j) \|_{X_p} := \mathcal{E} \| \sum_j \varepsilon_j \lambda_j u_j \|_p + \mathcal{E} \| \sum_j \varepsilon_j (-\Delta)^m u_j \|_p < \infty \}$$

and $Y_p := \text{Rad}(L_p)$ with norm $\|(f_j)\|_{Y_p} := \mathcal{E} \| \sum_j \varepsilon_j f_j \|_p$. For elementary properties of $\text{Rad}(X)$ we refer to [14]. Since $L_p$-spaces have non-trivial type we have $[\text{Rad}(L_p), \text{Rad}(L_q)]_\theta = \text{Rad}(L_r)$ for $1/r = (1 - \theta)/p + \theta/q$ (cf. [11]), and $(Y_p)$ is a complex interpolation scale. A similar argument shows that $(X_p)$ is a complex interpolation scale. Now we observe

$$\mathcal{E} \| \sum_j \varepsilon_j (\lambda_j + A) u_j \|_p \leq \mathcal{E} \| \sum_j \varepsilon_j \lambda_j u_j \|_p + \mathcal{E} \| A \sum_j \varepsilon_j u_j \|_p \leq \mathcal{E} \| \sum_j \varepsilon_j \lambda_j u_j \|_p + C \mathcal{E} \| \sum_j (-\Delta)^m u_j \|_p$$

where $C$ is the same constant as before. Thus Lemma 7 yields the claim for $K \in \{P^R, R\}$.

The case $\nu > 0$ is treated very similar. We just replace $A$ by $\nu + A$ and $(-\Delta)^m$ by $1 + (-\Delta)^m$.

Observe, e.g., that $P_{\theta, \nu, p}(a) < \infty$ and $\nu + A$ has a bounded inverse $L_p \to L_p$ if and only if there is a $C > 0$ such that

$$\|\lambda u\|_p + \| (1 + (-\Delta)^m) u \|_p \leq C \|(\lambda + \nu + A) u \|_p, \quad u \in W^{2m}_p.$$

For $k \in \{M, R, P^R\}$ the situation is similar.

\[ \square \]

**References**


