Contractivity of the H^∞-calculus and Blaschke products

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Abstract. It is well known that a densely defined operator A on a Hilbert space is accretive if and only if A has a contractive H^∞-calculus for any angle bigger than $\frac{\pi}{2}$. A third equivalent condition is that $\|(A-w)(A+\overline{w})^{-1}\| \leq 1$ for all $\operatorname{Re} w \geq 0$. In the Banach space setting, accretivity does not imply the boundedness of the H^∞-calculus any more. However, we show in this note that the last condition is still equivalent to the contractivity of the H^∞-calculus in all Banach spaces. Furthermore, we give a sufficient condition for the contractivity of the H^∞-calculus on $\mathbb{C}_+$, thereby extending a Hilbert space result of Sz.-Nagy and Foiaş to the Banach space setting.

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1. Introduction

It is well known that the Cayley transform $A \mapsto T = (A-1)(A+1)^{-1}$ provides, in a Hilbert space, a one-to-one correspondence between accretive operators A (i.e. negative generators of contractive semigroups) and bounded contractive operators T which do not have 1 as an eigenvalue (cf. [NF] theorem 4.1 in IV.4). If $\theta > \frac{\pi}{2}$, then this, combined with von Neumann's inequality for contractions, can be used to construct a contractive $H^\infty(\Sigma_\theta)$-calculus (see section 2 for the definition) for any accretive operator A (cf. [ADM]).

In a Banach space setting, this correspondence between accretive operators and contractions breaks down and von Neumann's inequality does not necessarily hold [Foi]. Thus in order to construct a contractive H^∞-calculus for Banach space operators we need stronger assumptions than accretivity. In this note we take a hint from one of the many known proofs of von Neumann's inequality (see [Dru]) and use classical approximations of bounded analytic functions on $\Sigma_{\frac{\pi}{2}}$ or $\mathbb{D}$ by Blaschke products. This enables us to connect the holomorphic functional calculus of a negative generator A of a semigroup and its Cayley transform T, thereby proving its contractivity.

In section 4 we show that a sectorial operator A of type $\frac{\pi}{2}$ on a Banach space X has a contractive $H^\infty(\Sigma_\theta)$-calculus for all $\theta > \frac{\pi}{2}$ if and only if for all $\lambda \in \Sigma_{\frac{\pi}{2}}$ and $x \in D(A)$

$$\|(A-\lambda)x\| \leq \|(A+\overline{\lambda})x\|. \tag{1.1}$$

Condition (1.1) is stronger than accretivity in a general Banach space X but it is easily seen that it is equivalent to accretivity if X is a Hilbert space. Hence our result extends the Hilbert space result quoted earlier and provides a relatively short proof for it. In theorem 4.4, we extend this functional calculus to functions in $H^\infty(\Sigma_{\frac{\pi}{2}})$ with a continuous boundary function on $i\mathbb{R} \cup \{\infty\}$, in the case that A is only a $\frac{\pi}{2}$-sectorial operator.

In general, (1.1) does not guarantee a bounded $H^\infty(\Sigma_{\frac{\pi}{2}})$-calculus, even in a Hilbert space. However, in section 5, we show that if we assume in addition that there exists a dense subset D of X such that

$$\int_0^\infty |\langle e^{-tA}x, x'\rangle|^2 dt \leq C_x \|x'\|^2 \text{ for } x \in D,\ x' \in X', \tag{1.2}$$

then A has a contractive $H^\infty(\Sigma_{\frac{\pi}{2}})$-calculus. Condition (1.2) is of importance in scattering theory and, in the Hilbert space setting, was studied by E.B. Davies in [Da1] and [Da2]. He shows in [Da1], theorem 6.26, that a completely non-unitary semigroup on a Hilbert space satisfies (1.2). Therefore our result generalizes the known fact that a completely non-unitary semigroup has an $H^\infty(\Sigma_{\frac{\pi}{2}})$ functional calculus, which can be derived from [NF], section III.8. We derive our Banach space result via the Cayley transform from a corresponding result for contractions T on X (see corollary 5.3) where we assume that

$$\sum_{n=0}^\infty |\langle T^n x, x'\rangle|^2 \leq C_x \|x'\|^2 \text{ for } x \in D,\ x' \in X'. \tag{1.3}$$

Clearly, every sectorial operator A on a Banach space X with a bounded $H^\infty(\Sigma_\theta)$-calculus has a contractive functional calculus in the equivalent norm

$$|\!|\!| x |\!|\!| = \sup\{\|\phi(A)x\| : \phi \in H^\infty(\Sigma_\theta),\ |\phi(\lambda)| \leq 1 \text{ for } \lambda \in \Sigma_\theta\}.$$

However, in section 6 we show that for many common examples of semigroup generators, $|\!|\!| \cdot |\!|\!|$ is not the natural norm of X.

In section 2 and 3, we recall some facts on the H^∞-calculus and Blaschke products that are essential for our argument.

2. Preliminaries on the H^∞-calculus

Notation. Let $\theta \in (0, \pi)$. As usual we shall set

$$\Sigma_\theta = \{re^{i\phi} : r > 0,\ |\phi| < \theta\}$$

and

$$H^\infty(\Sigma_\theta) = \{f : \Sigma_\theta \to \mathbb{C} : \ f \text{ is analytic and bounded}\}.$$

This space is complete when equipped with the norm $\|f\|_{\infty,\theta} = \sup_{\lambda\in\Sigma_\theta} |f(\lambda)|$. Put

$$H_0^\infty(\Sigma_\theta) = \{f \in H^\infty(\Sigma_\theta) : \ \exists\, \epsilon, C > 0 \text{ s.th. } |f(\lambda)| \le C \min(|\lambda|^\epsilon, |\lambda|^{-\epsilon})\}$$

and

$$R(\Sigma_\theta) = \operatorname{span}(H_0^\infty(\Sigma_\theta) \cup \{1, (1 + (\cdot))^{-1}\}).$$

Note that $R(\Sigma_\theta)$ is a subalgebra of $H^\infty(\Sigma_\theta)$ which contains all rational functions of non-positive degree and poles outside Σ_θ.
Let X be a Banach space and $\theta \in (0, \pi)$. An operator $A : D(A) \subseteq X \to X$ is called *θ-sectorial,* if

1. $D(A)$ is dense in X.
2. The spectrum $\sigma(A)$ is contained in $\overline{\Sigma_\theta}$.
3. For all $\omega > \theta$ there is a $C_\omega > 0$ such that $\|\lambda R(\lambda, A)\| \le C_\omega$ for all $\lambda \in \overline{\Sigma_\omega}^c$.

For such an operator, one can construct for every $\omega > \theta$ a linear and multiplicative mapping $\Phi_\omega : R(\Sigma_\omega) \longrightarrow B(X)$ such that for $\mu \in \mathbb{C}\backslash\overline{\Sigma_\omega}$ and $f \in H_0^\infty(\Sigma_\omega)$

$$\Phi_\omega(1) = \mathrm{Id}_X,\ \Phi_\omega((\mu - (\cdot))^{-1}) = (\mu - A)^{-1} \text{ and } \Phi_\omega(f) = \frac{1}{2\pi i}\int_{\partial\Sigma_{(\omega+\theta)/2}} f(\lambda)R(\lambda, A)d\lambda.$$

We call Φ_ω the $H^\infty(\Sigma_\omega)$-calculus of A. For $\omega_1 > \omega_2$, Φ_{ω_1} and Φ_{ω_2} coincide on $R(\Sigma_{\omega_1})$. If in addition A has dense range then Φ_ω can be extended to

$$\begin{aligned} H_A^\infty(\Sigma_\omega) = \{f \in H^\infty(\Sigma_\omega) : \ &\exists\, (f_n)_n \subseteq R(\Sigma_\omega) \text{ s.th.}\\ &f_n \to f \text{ pointwise and } \sup_n \|f_n(A)\| + \|f_n\|_{\infty,\omega} < \infty\}.\end{aligned}$$

For $f \in H_A^\infty(\Sigma_\omega)$ and associated $(f_n)_n \subseteq R(\Sigma_\omega)$, $\Phi_\omega(f)x = \lim_n \Phi_\omega(f_n)x$ for all $x \in X$. This implies clearly that $\|\Phi_\omega(f)\| \le \liminf_n \|\Phi_\omega(f_n)\|$. The map

$$\Phi_\omega : H_A^\infty(\Sigma_\omega) \longrightarrow B(X)$$

is still linear and multiplicative. Further, $H_A^\infty(\Sigma_\omega)$ equals $H^\infty(\Sigma_\omega)$ iff there exists $C > 0$ such that for all $f \in H_0^\infty(\Sigma_\omega)$ one has $\|f(A)\| \le C\|f\|_{\infty,\omega}$. In this case, we say that A has a *bounded $H^\infty(\Sigma_\omega)$-calculus.* For more information on the H^∞-calculus, see for example [CDMY], especially section 2 and [KW], especially chapter 9. We denote $\Phi_\omega(f)$ by $f(A)$.

3. Preliminaries on Blaschke products

In this section, we develop the necessary background on Blaschke products (see also [Gar] for more information on this topic).

Notation. We put $\mathbb{D} = \{z \in \mathbb{C} : \, |z| < 1\}$ and consider the two spaces

$$H^\infty(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} \text{ analytic and bounded}\}$$

and

$$A(\mathbb{D}) = \{f \in H^\infty(\mathbb{D}) : \, f \text{ has a continuous extension to } \overline{\mathbb{D}}\}.$$

They are equipped with the norm $\|f\|_{\infty,\mathbb{D}} = \|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$, for which they are complete. For $\mu \in \mathbb{D}$, we define

$$f_\mu(z) = \frac{z-\mu}{1-\overline{\mu}z}.$$

This function is called a *Blaschke factor on* $\mathbb{D}$ and is clearly analytic on the neighborhood $\frac{1}{\mu}\mathbb{D}$ of $\mathbb{D}$. For $\mu \in \mathbb{D}$, $f_\mu(\partial\,\mathbb{D}) \subseteq \partial\,\mathbb{D}$, since for $\theta \in \mathbb{R}$

$$|f_\mu(e^{i\theta})| = \left|\frac{e^{i\theta}-\mu}{1-\overline{\mu}e^{i\theta}}\right| = \left|e^{i\theta}\frac{1-\mu e^{-i\theta}}{1-\overline{\mu e^{-i\theta}}}\right| = 1.$$

Thus by the maximum principle, $\|f_\mu\|_\infty = 1$.
A function $f \in A(\mathbb{D})$ is called a *(finite) Blaschke product on* $\mathbb{D}$ if it is of the form

$$f(z) \equiv e^{i\theta} \quad \text{or} \quad f(z) = e^{i\theta}\prod_{k=1}^n f_{\mu_k}(z)$$

for some $n \in \mathbb{N}$, $\mu_k \in \mathbb{D}$ and $\theta \in \mathbb{R}$. A function $f \in H^\infty(\mathbb{D})$ is a finite Blaschke product if and only if

1. f is analytic on some $a\,\mathbb{D}$ with $a > 1$.
2. $f(\partial\,\mathbb{D}) \subseteq \partial\,\mathbb{D}$.

Indeed, a finite Blaschke product clearly has the two properties. Suppose now that f satisfies 1. and 2. If f had infinitely many zeros in $\mathbb{D}$ then there would exist an accumulation point of them in $\overline{\mathbb{D}}$. By 1. and the identity theorem, $f \equiv 0$, which contradicts 2. Hence f has only finitely many zeros and there exists a finite Blaschke product b which has the same zeros counted with their multiplicity. Then f/b and b/f also satisfy 1. and 2. Therefore, for $z \in \mathbb{D}$, $|f(z)/b(z)| \leq 1$ and $|b(z)/f(z)| \leq 1$, and thus $|f(z)/b(z)| = 1$, which implies that $f(z)/b(z)$ is constant. Hence $f(z) = e^{i\theta}b(z)$ for some $\theta \in \mathbb{R}$.
If $T \in B(X)$ with spectrum $\sigma(T) \subseteq \overline{\mathbb{D}}$, then for $f(z) = \sum a_n z^n$ holomorphic in $a\,\mathbb{D}$ for some $a > 1$, we define $f(T) = \sum a_n T^n$, which is the Dunford calculus of T. In particular, $f_\mu(T) = (T-\mu)(1-\overline{\mu}T)^{-1}$.
We also need the Blaschke factors on $\Sigma_{\frac{\pi}{2}}$. We define the conformal mappings

$$\zeta : \mathbb{D} \to \Sigma_{\frac{\pi}{2}},\, z \mapsto -\frac{z+1}{z-1} \quad \text{and} \quad \tau = \zeta^{-1} : \Sigma_{\frac{\pi}{2}} \to \mathbb{D},\, \lambda \mapsto \frac{\lambda-1}{\lambda+1}.$$

Then $H^\infty(\Sigma_{\frac{\pi}{2}}) = \{f\circ\tau : f \in H^\infty(\mathbb{D})\}$. We put $A(\Sigma_{\frac{\pi}{2}}) = \{f\circ\tau : f \in A(\mathbb{D})\} = \{f \in H^\infty(\Sigma_{\frac{\pi}{2}}) : \, f$ has a continuous extension to $\overline{\Sigma_{\frac{\pi}{2}}}$ and at $\infty\}$. The spaces $H^\infty(\Sigma_{\frac{\pi}{2}})$

and $A(\Sigma_{\frac{\pi}{2}})$ are again equipped with the infinity norm. Note that for $\theta > \frac{\pi}{2}$, $H^\infty(\Sigma_\theta) \not\subset A(\Sigma_{\frac{\pi}{2}})$, but $R(\Sigma_\theta) \subset A(\Sigma_{\frac{\pi}{2}})$. For $w \in \Sigma_{\frac{\pi}{2}}$, we define

$$b_w(\lambda) = \frac{\lambda - w}{\lambda + \overline{w}}, \quad \lambda \in \Sigma_{\frac{\pi}{2}}$$

which we call a *Blaschke factor on* $\Sigma_{\frac{\pi}{2}}$. For $\mu \in \mathbb{D}$, the Blaschke factor f_μ on $\mathbb{D}$ is related to the Blaschke factor $b_{\zeta(\mu)}$ on $\Sigma_{\frac{\pi}{2}}$ by the identity

$$f_\mu \circ \tau = f_\mu(1) \cdot b_{\zeta(\mu)}.$$

As above we call $f \in A(\Sigma_{\frac{\pi}{2}})$ a *(finite) Blaschke product on* $\Sigma_{\frac{\pi}{2}}$, if it is of the form

$$f(\lambda) \equiv e^{i\theta} \quad \text{or} \quad f(\lambda) = e^{i\theta} \prod_{k=1}^{n} b_{w_k}(\lambda)$$

for some $n \in \mathbb{N}$, $w_k \in \Sigma_{\frac{\pi}{2}}$ and $\theta \in \mathbb{R}$. We denote by $B_{\mathbb{D}}$ (resp. $B_{\Sigma_{\frac{\pi}{2}}}$) the set of finite Blaschke products, which is contained in the unit ball of $A(\mathbb{D})$ (resp. $A(\Sigma_{\frac{\pi}{2}})$).

The next lemma and its proof is essentially taken from [Gar]. Since these results are essential for us we include a proof for the convenience of the reader.

Lemma 3.1.

1. *(Carathéodory). For every f in the unit ball of $H^\infty(\mathbb{D})$ (resp. $H^\infty(\Sigma_{\frac{\pi}{2}})$), there exists a sequence in $B_{\mathbb{D}}$ (resp. $B_{\Sigma_{\frac{\pi}{2}}}$) which converges pointwise on $\mathbb{D}$ (resp. $\Sigma_{\frac{\pi}{2}}$) to f.*
2. *(Bernard). $\operatorname{co} B_{\mathbb{D}}$, the convex hull of $B_{\mathbb{D}}$, is norm dense in the unit ball of $A(\mathbb{D})$. Hence also $\operatorname{co} B_{\Sigma_{\frac{\pi}{2}}}$ is norm dense in the unit ball of $A(\Sigma_{\frac{\pi}{2}})$.*

Proof. 1. It clearly suffices to prove the statement for $\mathbb{D}$. Write $f(z) = \sum_{k=0}^\infty c_k z^k$. It suffices to find for every $n \in \mathbb{N}$ a finite Blaschke product $B_{n,f}(z) = \sum_{k=0}^\infty d_k z^k$ such that $d_k = c_k$ for $k \leq n-1$. Indeed, by the Cauchy integral formula, $|c_k|, |d_k| \leq 1$ for all $k \in \mathbb{N}$. Then for $|z| < 1$ fixed,

$$|f(z) - B_{n,f}(z)| \leq \sum_{k=n}^{\infty} (|c_k| + |d_k|)|z|^k \leq 2|z|^n (1 - |z|)^{-1} \to 0.$$

If $|c_0| = 1$, then by the maximum principle, f is constant and $B_{n,f}(z) \equiv c_0$ suffices. So suppose from now on that $|c_0| < 1$. We proceed by induction. For $n = 1$, $B_{1,f}(z) = f_{-c_0}(z) = \frac{z+c_0}{1+\overline{c_0}z}$ suffices. Suppose that for a given $n \in \mathbb{N}$ and all h in the unit ball of $H^\infty(\mathbb{D})$, there exists $B_{n-1,h}$ such that $B_{n-1,h} - h$ has a zero of multiplicity $n-1$ at 0. Let

$$h(z) = \frac{1}{z} \frac{f(z) - c_0}{1 - \overline{c_0} f(z)} = \frac{1}{z} f_{c_0}(f(z)).$$

If $|z| < 1$ then for all $r \in (|z|, 1)$ we have $|h(z)| \leq \frac{1}{r} |f_{c_0}(f(z))| \leq \frac{1}{r}$ and hence h is in the unit ball of $H^\infty(\mathbb{D})$. We will construct $B_n = B_{n,f}$ by means of $B_{n-1} = B_{n-1,h}$. Put

$$B_n(z) = \frac{zB_{n-1}(z) + c_0}{1 + \overline{c_0} z B_{n-1}(z)} = f_{-c_0}(zB_{n-1}(z)).$$

Then B_n is analytic on $a\,\mathbb{D}$ for some $a > 1$ and satisfies $B_n(\partial\,\mathbb{D}) \subseteq \partial\,\mathbb{D}$. Hence B_n is a finite Blaschke product.

$$f(z) - B_n(z) = f_{-c_0}(zh(z)) - f_{-c_0}(zB_{n-1}(z)) = \frac{(1-|c_0|^2)z(h(z) - B_{n-1}(z))}{(1+\overline{c_0}zh(z))(1+\overline{c_0}zB_{n-1}(z))}.$$

Here we see that $f - B_n$ has a zero of multiplicity n at 0. The induction is complete.
2. Let $p = \sum_{n=0}^{N} a_n z^n$ be a polynomial in $A(\mathbb{D})$ with $\|p\|_\infty < 1$. It clearly suffices to show that such a p can be uniformly approximated by a finite convex combination of finite Blaschke products. Let $q(z) = \sum_{n=0}^{N} \overline{a_n} z^{N-n} = z^N \overline{p(z)}$, so that $q \in A(\mathbb{D})$ and $|q(z)| < 1$ on $\overline{\mathbb{D}}$. For $z, \xi \in \overline{\mathbb{D}}$ put

$$r(z,\xi) = \frac{p(z) + \xi z^N}{1 + \xi q(z)} = f_{-p(z)}(\xi z^N).$$

For every $\xi \in \overline{\mathbb{D}}$, $r(\cdot,\xi)$ can be analytically continued on $a\,\mathbb{D}$ for some $a > 1$. Also for every $z \in \overline{\mathbb{D}}$, $r(z,\cdot)$ can be analytically continued on $\|q\|_\infty^{-1}\,\mathbb{D}$. Since $|r(z,e^{it})| = |f_{-p(z)}(e^{it}z^N)| = 1$ for all $z \in \partial\,\mathbb{D}$, $r(\cdot,e^{it})$ is a finite Blaschke product. By the mean value property for $r(z,\cdot)$,

$$p(z) = r(z,0) = \int_0^{2\pi} r(z,e^{it})\frac{dt}{2\pi},$$

where $[0,2\pi] \ni t \mapsto r(z,e^{it}) \in A(\mathbb{D})$ is continuous. So p can be approximated in $A(\mathbb{D})$ by a sequence of finite convex combinations of finite Blaschke products. □

4. Contractivity of the H^∞-calculus

Our result is motivated by the following observation about Hilbert space operators.

Lemma 4.1. *Let H be a Hilbert space and let $A : D(A) \subseteq H \to H$ be an operator. The following are equivalent:*

1. *A is accretive, i.e. for all $x \in D(A)$ we have* $\operatorname{Re}\langle Ax, x\rangle \geq 0$.
2. *For all $w \in \mathbb{C}$ with* $\operatorname{Re} w > 0$ *and all $x \in D(A)$,* $\|(A-w)x\| \leq \|(A+\overline{w})x\|$.
3. *There exists a $w \in \mathbb{C}$ with* $\operatorname{Re} w > 0$ *such that for all $x \in D(A)$,* $\|(A-w)x\| \leq \|(A+\overline{w})x\|$.
4. *For every finite Blaschke product $b \in B_{\Sigma_{\frac{\pi}{2}}}$,* $\|b(A)\| \leq 1$.

Proof. The following inequalities are equivalent for $x \in D(A)$:

$$\|(A-w)x\| \leq \|(A+\overline{w})x\|$$
$$\langle (A-w)x, (A-w)x\rangle \leq \langle (A+\overline{w})x, (A+\overline{w})x\rangle$$
$$\langle Ax, Ax\rangle - 2\operatorname{Re}[\overline{w}\langle Ax, x\rangle] + |w|^2\langle x, x\rangle \leq \langle Ax, Ax\rangle + 2\operatorname{Re}[w\langle Ax, x\rangle] + |w|^2\langle x, x\rangle$$
$$0 \leq \operatorname{Re}[(w+\overline{w})\langle Ax, x\rangle]$$
$$0 \leq \operatorname{Re}\langle Ax, x\rangle.$$

So if 3. holds for some $w \in \Sigma_{\frac{\pi}{2}}$, then 1. holds, which in turn implies that 2. holds. That 2. implies 3. is evident and the equivalence of 2. and 4. holds since for every $w \in \Sigma_{\frac{\pi}{2}}$, $A + \overline{w}$ is a bijection $D(A) \to X$. □

In a general Banach space, the equivalence of 1. and 4. above breaks down. As was shown in [KW, section 10], there exist accretive operators without a bounded H^∞-calculus. Condition 2. above is however still equivalent to the boundedness of the H^∞-calculus in the Banach space context.

Theorem 4.2. *Let X be a Banach space, $\theta < \frac{\pi}{2}$ and $A : D(A) \subseteq X \to X$ be θ-sectorial with dense range. Then A satisfies*

$$\|(A - w)x\| \le \|(A + \overline{w})x\| \text{ for all } \operatorname{Re} w > 0,\ x \in D(A) \tag{4.1}$$

if and only if A has a bounded $H^\infty(\Sigma_{\frac{\pi}{2}})$-calculus and

$$\|f(A)\| \le \|f\|_{\infty,\frac{\pi}{2}} \quad (f \in H^\infty(\Sigma_{\frac{\pi}{2}})).$$

Proof. The condition (4.1) implies that $b_w(A) = (A-w)(A+\overline{w})^{-1}$ is a contraction. Thus, $b(A)$ is a contraction for every finite Blaschke product b. Suppose now that $f \in H^\infty(\Sigma_{\frac{\pi}{2}})$ with $\|f\|_{\infty,\frac{\pi}{2}} \le 1$. Then by lemma 3.1, there exists a sequence f_n of finite Blaschke products converging to f pointwise on $\Sigma_{\frac{\pi}{2}}$. Since $\|f_n\|, \|f_n(A)\| \le 1$ for all $n \in \mathbb{N}$, f belongs to $H^\infty_A(\Sigma_{\frac{\pi}{2}})$ and $\|f(A)\| \le \liminf_n \|f_n(A)\| \le 1$. The necessity of (4.1) is clear, since it is equivalent to $\|b_w(A)\| \le \|b_w\|_{\infty,\frac{\pi}{2}} = 1$ for all $\operatorname{Re} w > 0$. □

Of course, the last proof also shows that the boundedness of the $H^\infty(\Sigma_{\frac{\pi}{2}})$-calculus in the situation of theorem 4.2 can be characterized by the uniform boundedness of the Blaschke products:

$$\|b(A)\| \le C \text{ for some } C < \infty \text{ and every finite Blaschke product } b \text{ on } \Sigma_{\frac{\pi}{2}}. \tag{4.2}$$

If one compares the boundedness of the H^∞-calculus to the boundedness of the semigroup, then (4.2) corresponds to the Hille-Yosida condition $\|\lambda^n(\lambda+A)^n\| \le C$ for all $\lambda > 0$ and (4.1) corresponds to the accretivity of the operator A. Note that if the $H^\infty(\Sigma_{\frac{\pi}{2}})$-calculus of A is bounded, then it is always contractive in an equivalent norm on X, given for example by

$$|\!|\!| x |\!|\!| = \sup\{\|f(A)x\| : f \in H^\infty(\Sigma_{\frac{\pi}{2}}) : \|f\|_{\infty,\frac{\pi}{2}} \le 1\}.$$

Since θ-sectoriality for $\theta < \frac{\pi}{2}$ already implies that $-A$ generates an analytic semigroup, it is worthwhile to state in addition a somewhat weaker result for $\frac{\pi}{2}$-sectorial operators which covers all negative generators of a bounded semigroup. To this end we need the following lemma.

Lemma 4.3. *Let $A : D(A) \subseteq X \to X$ be $\frac{\pi}{2}$-sectorial. Suppose that* (4.1) *holds. For $r \in (0, 1)$, let*

$$A_r = (1 - r + A(1 + r))(1 + r + A(1 - r))^{-1} = \zeta \circ \psi_r \circ \tau(A) \in B(X),$$

where ζ and τ are the conformal mappings as in section 3 *and $\psi_r : \mathbb{D} \to \mathbb{D}$, $z \mapsto rz$. Then*

1. $\sigma(A_r) \subset \Sigma_{\frac{\pi}{2}}$.
2. A_r *is again* $\frac{\pi}{2}$*-sectorial, and this uniformly in* r*, in the sense that for all* $\omega \in (\frac{\pi}{2}, \pi)$ *there exists* $C_\omega > 0$ *such that for all* $r \in (0,1)$ *and* $\lambda \in \mathbb{C}\backslash\overline{\Sigma_\omega}$ *we have* $\|\lambda R(\lambda, A_r)\| \leq C_\omega$.
3. *For* $\operatorname{Re}\lambda < 0$, $R(\lambda, A_r) \to R(\lambda, A)$ *in* $B(X)$ *as* $r \to 1$.
4. (4.1) *holds for* A_r *in place of* A.

Proof. 1. By (4.1), $\tau(A)$ is a contraction, and hence $\sigma(\psi_r \circ \tau(A)) \subset \mathbb{D}$. By the spectral mapping theorem, $\sigma(A_r) = \sigma(\zeta \circ \psi_r \circ \tau(A)) \subset \Sigma_{\frac{\pi}{2}}$.
2. Fix some $\omega \in (\frac{\pi}{2}, \pi)$. Let $\lambda \in \mathbb{C}\backslash\overline{\Sigma_\omega}$. Then $\lambda(\lambda - A_r)^{-1}$

$$= \{\lambda(1+r) + A\lambda(1-r)\} \left[A(\lambda(1-r) - (1+r)) + \lambda(1+r) - (1-r)\right]^{-1}. \quad (4.3)$$

We split this expression into two summands (I) and (II).

$$\begin{aligned}
(\mathrm{I}) &= \lambda(1+r)[\ldots]^{-1} \\
&= \frac{\lambda(1+r)}{\lambda(1-r) - (1+r)} \left[A + \frac{\lambda(1+r) - (1-r)}{\lambda(1-r) - (1+r)}\right]^{-1} \\
&= \frac{\lambda(1+r)}{\lambda(1+r) - (1-r)} \mu(A+\mu)^{-1},
\end{aligned}$$

with $\mu = \frac{\lambda(1+r)-(1-r)}{\lambda(1-r)-(1+r)}$. Note that $\left|\frac{\lambda(1+r)}{\lambda(1+r)-(1-r)}\right| \leq 1$. Further, $\mu = \frac{\lambda - a^{-1}}{\lambda - a} a$ with $a = \frac{1+r}{1-r} > 1$, so $|\arg\mu| = |\arg(\lambda - a^{-1}) - \arg(\lambda - a)| \leq |\arg(\lambda - a^{-1})| < \pi - \omega$, and thus $\|\mu(A+\mu)^{-1}\| \leq C_\omega = \sup\{\|\nu R(\nu, A)\| :\ \nu \in \mathbb{C}\backslash\overline{\Sigma_\omega}\}$.

$$(\mathrm{II}) = A\lambda(1-r)[\ldots]^{-1} = \frac{\lambda(1-r)}{\lambda(1-r) - (1+r)} A[A+\mu]^{-1}.$$

Now $\left|\frac{\lambda(1-r)}{\lambda(1-r)-(1+r)}\right| \leq 1$. Further, $\|A(A+\mu)^{-1}\| \leq 1 + \|\mu(A+\mu)^{-1}\| \leq 1 + C_\omega$, since $\mu \in \Sigma_{\pi-\omega}$.
3. This can be deduced using (4.3) and the continuity of the map $\lambda \mapsto AR(\lambda, A)$ from $\mathbb{C}\backslash\overline{\Sigma_\omega}$ to $B(X)$.
4. Let $\operatorname{Re} w > 0$, put $\mu = \tau(w) \in \mathbb{D}$ and $T = \tau(A)$. $b_w(A_r) = b_w \circ \zeta \circ \psi_r \circ \tau(A) = f_\mu(1)^{-1} f_\mu(rT)$. We show that $f_\mu(rT) = (rT - \mu)(1 - \overline{\mu} rT)^{-1}$ is a contraction, which implies that $b_w(A_r)$ is one.

$$f_\mu(z) = \frac{z - \mu}{1 - \overline{\mu} z} = -\mu + \frac{1 - |\mu|^2}{\overline{\mu}} \frac{\overline{\mu} z}{1 - \overline{\mu} z}.$$

This shows that $f_\mu(rz) = \alpha f_{r\mu}(z) + \beta$ with $\alpha = \frac{(1-|\mu|^2)r}{1-|r\mu|^2}$ and $\beta = \mu r \frac{(1-|\mu|^2)r}{1-|r\mu|^2} - \mu$. So, $f_\mu(rT) = \alpha f_{r\mu}(T) + \beta \operatorname{Id}_X$. Now $f_{r\mu}(T) = f_{r\mu}(1) b_{\zeta(r\mu)}(A)$ is a contraction, so that it suffices to show that $|\alpha| + |\beta| \le 1$.

$$\begin{aligned} |\alpha| + |\beta| &= \frac{(1-|\mu|^2)r}{1-|r\mu|^2} + |\mu| \left| \frac{r^2(1-|\mu|^2)}{1-|r\mu|^2} - 1 \right| \\ &= \frac{(1-|\mu|^2)r}{1-|r\mu|^2} + |\mu| \frac{r^2-1}{1-|r\mu|^2} \\ &= \frac{(1-a^2)r + a(1-r^2)}{1-a^2r^2}, \end{aligned}$$

where $a = |\mu| \in (0,1)$. So $|\alpha| + |\beta| \le 1$ iff $r^2(a-a^2) + r(a^2-1) + 1 - a \ge 0$. The left hand side of the last inequality equals 0 for $r = 1$, and its derivative with respect to r is $2r(a-a^2) + a^2 - 1 = 2(a-a^2)(r-1) - (a-1)^2 \le 0$ for $r \le 1$. So the inequality is indeed fulfilled. □

Theorem 4.4. *Let $A : D(A) \subseteq X \to X$ be $\frac{\pi}{2}$-sectorial. Suppose that*

$$\|(A-w)x\| \le \|(A+\overline{w})x\| \text{ for all } w \in \Sigma_{\frac{\pi}{2}} \text{ and } x \in D(A).$$

Then for all $\theta > \frac{\pi}{2}$ and all $f \in H_0^\infty(\Sigma_\theta)$,

$$\|f(A)\| \le \|f\|_{\infty,\frac{\pi}{2}},$$

and if A has dense range, this holds for all $f \in H^\infty(\Sigma_\theta)$. Further, there exists a unique contractive algebra homomorphism $\Phi : A(\Sigma_{\frac{\pi}{2}}) \to B(X)$, which coincides with the H^∞-calculus for A on $\bigcup_{\theta>\frac{\pi}{2}} R(\Sigma_\theta)$.

Proof. For $r \in (0,1)$, let A_r be as in lemma 4.3. By lemma 4.3 (4), $\|b_w(A_r)\| \le 1$ for all $w \in \Sigma_{\frac{\pi}{2}}$. Therefore, $f(A_r)$ is a contraction for all $f \in B_{\Sigma_{\frac{\pi}{2}}}$ and thus for all $f \in \operatorname{co} B_{\Sigma_{\frac{\pi}{2}}}$. Let $f \in A(\Sigma_{\frac{\pi}{2}})$ such that $\|f\|_{\infty,\frac{\pi}{2}} \le 1$. By lemma 3.1, there exists a sequence $(f_n)_n \subset \operatorname{co} B_{\Sigma_{\frac{\pi}{2}}}$ with $\|f_n - f\|_{\infty,\frac{\pi}{2}} \to 0$. Since $A_r \in B(X)$ with $\sigma(A_r) \subset \Sigma_{\frac{\pi}{2}}$, by the Dunford calculus $f_n(A_r) \to f(A_r)$ and hence $\|f(A_r)\| \le 1$. (The a priori continuity of $A(\Sigma_{\frac{\pi}{2}}) \to B(X)$, $f \mapsto f(A_r)$ was the reason for the introduction of the A_r.) If now $f = f_0 + a + b(1+(\cdot))^{-1} \in R(\Sigma_\theta)$ for some $\theta > \omega > \frac{\pi}{2}$, then for $r \to 1 : f(A_r) = (2\pi i)^{-1} \int_{\partial\Sigma_\omega} f(\lambda)R(\lambda, A_r)d\lambda + a \operatorname{Id}_X + b(1+A_r)^{-1} \to (2\pi i)^{-1} \int_{\partial\Sigma_\omega} f(\lambda)R(\lambda, A)d\lambda + a \operatorname{Id}_X + b(1+A)^{-1} = f(A)$ by the convergence of the resolvents (lemma 4.3 (3)), the uniform sectoriality (lemma 4.3 (2)) and Lebesgue's theorem. This shows $\|f(A)\| \le \|f\|_{\infty,\frac{\pi}{2}}$. Existence and uniqueness of Φ now follow from the density of $\bigcup_{\theta>\frac{\pi}{2}} R(\Sigma_\theta)$ in $A(\Sigma_{\frac{\pi}{2}})$.
If A has dense range, then for $f \in H^\infty(\Sigma_\theta)$, $\|f(A)\| \le \liminf_n \|f\rho_n\|_{\infty,\frac{\pi}{2}} = \|f\|_{\infty,\frac{\pi}{2}}$, where $\rho_n(\lambda) = \left(\lambda(1+\lambda)^{-2}\right)^{\frac{1}{n}}$. □

Combining theorems 4.2 and 4.4 with lemma 4.1 enables one to deduce the following well-known Hilbert space result. However, our proof is different from the ones given in [NF] or [ADM].

Corollary 4.5. *Let H be a Hilbert space. Suppose that A is a θ-sectorial operator on H with dense range such that A is accretive.*

1. *If $\theta < \frac{\pi}{2}$, then A has a contractive H^∞-calculus for the angle $\frac{\pi}{2}$.*
2. *If $\theta = \frac{\pi}{2}$, then A has a contractive H^∞-calculus for any angle $> \frac{\pi}{2}$.*

Remark 4.6. The key argument in both theorem 4.2 and 4.4 is to show first the contractivity of Blaschke operators and then use a density argument. We point out that in [vN], von Neumann uses a similar approach in the original proof of the inequality named after him.

5. Square integrable vectors and the calculus on $\Sigma_{\frac{\pi}{2}}$

It is not always possible to extend the holomorphic functional calculus for contractions in Hilbert spaces as constructed in [NF] to all of $H^\infty(\mathbb{D})$. Likewise, one cannot expect an $H^\infty(\Sigma_{\frac{\pi}{2}})$-calculus for all operators satisfying condition (4.1). In the Hilbert space case, it can be derived from [NF], sections III.2 and III.8, that an accretive operator A has an $H^\infty(\Sigma_{\frac{\pi}{2}})$- calculus if the unitary part of A (in the sense of [NF], III.8) has a uniformly continuous spectral measure. In particular, A has an $H^\infty(\Sigma_{\frac{\pi}{2}})$-calculus if it is completely non-unitary (see [NF] III.8). It is also known that in these circumstances A has a dense set D of "square integrable vectors" in the sense that

$$\int_0^\infty |\langle e^{-tA}x, x'\rangle|^2 dt < C_x \|x'\|^2 \text{ for all } x \in D \text{ and } x' \in X' \tag{5.1}$$

(see [Da1] section 6.5). In the Banach space setting, one has to find a replacement for the notions of "absolutely continuous spectral measure" and "completely non-unitary operators". Davies' result convinced us that condition (5.1) may be a good replacement for these notions. It also allows us to prove an extension of the Hilbert space result quoted above in the Banach space setting. But first we connect square integrable vectors of negative generators of semigroups to square summable vectors of its Cayley transform.

Lemma 5.1. *Let $-A$ be the generator of a c_0-semigroup e^{-tA} on X. Let $x' \in X'$, $x \in D(A)$ and $y = \frac{1}{2}(1 + A)x$. Suppose that $\int_0^\infty |\langle e^{-tA}y, x'\rangle|^2 dt < \infty$. Then for $T = (A - 1)(A + 1)^{-1} \in B(X)$:*

$$\sum_{n=0}^{\infty} |\langle T^n x, x'\rangle|^2 = 2 \int_0^\infty |\langle e^{-tA}y, x'\rangle|^2 dt.$$

Proof. Put $f(\lambda) = \langle(\lambda + \frac{1}{2}A)^{-1}y, x'\rangle$. Then $f(\lambda) = \int_0^\infty e^{-\lambda t}g(t)dt$ with $g(t) = \langle e^{-\frac{t}{2}A}y, x'\rangle$ and $\|g\|_{L^2(0,\infty)} < \infty$. Further, $f^{(k)}(\lambda) = k!(-1)^k\langle(\lambda + \frac{1}{2}A)^{-k-1}y, x'\rangle$

for $k \in \mathbb{N}_0$. Now for $n \in \mathbb{N}_0$

$$\begin{aligned}\langle T^n x, x'\rangle &= \langle [(A-1)(A+1)^{-1}]^n x, x'\rangle = \langle (\mathrm{Id}_X - 2(A+1)^{-1})^n x, x'\rangle \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \langle (\frac{1}{2} + \frac{1}{2}A)^{-k} x, x'\rangle \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k!} k! (-1)^k \langle (\frac{1}{2} + \frac{1}{2}A)^{-k-1} y, x'\rangle = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k!} f^{(k)}(\frac{1}{2}) =: q_n.\end{aligned}$$

By [Sho] (see also [Roo] theorem 1 for the case $\nu = 0$), the fact that f is the Laplace transform of a function in $L^2(0,\infty)$ implies that

$$2\int_0^\infty |\langle e^{-tA} y, x'\rangle|^2 dt = \int_0^\infty |g(t)|^2 dt = \sum_{n=0}^{\infty} |q_n|^2 = \sum_{n=0}^{\infty} |\langle T^n x, x'\rangle|^2.$$

□

Theorem 5.2. *Let X be a Banach space and $D \subset X$ a dense subset. Let A be a $\frac{\pi}{2}$-sectorial operator on X satisfying (4.1) and (5.1), i.e.*

1. $\|(A-w)x\| \le \|(A+\overline{w})x\|$ *for all* $x \in D(A)$ *and* $\mathrm{Re}\, w > 0$.
2. $\forall\, x \in D\, \exists\, C > 0\, \forall\, x' \in X' : \int_0^\infty |\langle e^{-tA} x, x'\rangle|^2 dt \le C\|x'\|^2$.

Then there is a unique extension of the $A(\Sigma_{\frac{\pi}{2}})$-calculus in theorem 4.4 to a linear, multiplicative and contractive $\Phi : H^\infty(\Sigma_{\frac{\pi}{2}}) \to B(X)$ with the following convergence property:
If $f, f_1, f_2, \ldots \in H^\infty(\Sigma_{\frac{\pi}{2}})$ with $\sup_{n\in\mathbb{N}} \|f_n\|_{\infty,\frac{\pi}{2}} < \infty$ and $f_n(\lambda) \to f(\lambda)$ for a.a. $\lambda \in i\mathbb{R}$, then $\Phi(f_n)x \to \Phi(f)x$ for all $x \in X$.

Proof. For $r \in (0,1)$ and $f \in H^\infty(\mathbb{D})$, we put $f_r(\lambda) = f(r\lambda)$, so that $f_r \in A(\mathbb{D})$. Let $T = (A-1)(A+1)^{-1}$. We claim:

$$\text{For } f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^\infty(\mathbb{D}) \text{ and } x \in X,\ f_r(T)x \text{ converges in } X \text{ for } r \to 1. \tag{5.2}$$

Let first $x \in (\frac{1}{2} + \frac{1}{2}A)^{-1}(D)$ and $x' \in X'$. By lemma 5.1, the sequence $\langle T^n x, x'\rangle$ is in l^2 with $\|(\langle T^n x, x'\rangle)_n\|_{l^2} \le C\|x'\|$. So for $r, s \in (0,1)$, $r > s$,

$$\begin{aligned}|\langle f_r(T)x - f_s(T)x, x'\rangle| &\le \sum_{n=0}^{\infty} |a_n (r^n - s^n) \langle T^n x, x'\rangle| \\ &\le (1 - \frac{s^N}{r^N}) \sum_{n=0}^{N} |a_n|\, |\langle T^n x, x'\rangle| + 2\|(a_n)_n\|_{l^2(\{N+1,N+2,\ldots\})}\, \|\,(\langle T^n x, x'\rangle)_n\,\|_{l^2}.\end{aligned}$$

Since the a_n are the Fourier coefficients of $f \in H^\infty(\mathbb{D}) \subset L^2(\partial\,\mathbb{D})$, we have $(a_n)_n \in l^2$. Now, for a given $\epsilon > 0$, choose now first N large, and then r, s sufficiently near to 1 in order to dominate the expression by ϵ, uniformly for $\|x'\| \le 1$. This gives (5.2) for $x \in (\frac{1}{2} + \frac{1}{2}A)^{-1}(D)$. If $x \in X$ is arbitrary, we can choose $x_\epsilon \in (\frac{1}{2} + \frac{1}{2}A)^{-1}(D)$ such that $\|x - x_\epsilon\| \le \epsilon$. Indeed, since D is dense in X and $(\frac{1}{2} + \frac{1}{2}A)^{-1}$ is an isomorphism $(X, \|\cdot\|_X) \to (D(A), \|\cdot\|_X + \|A\cdot\|_X)$, $(\frac{1}{2}+\frac{1}{2}A)^{-1}(D)$ is dense in $D(A)$ with respect to $\|\cdot\|_X + \|A\cdot\|_X$, and thus also with respect to $\|\cdot\|_X$. Finally, $D(A)$ is dense in X. Then

$$\begin{aligned}\|f_r(T)x - f_s(T)x\| &\leq \|f_r(T)(x-x_\epsilon)\| + \|f_s(T)(x-x_\epsilon)\| + \|f_r(T)x_\epsilon - f_s(T)x_\epsilon\| \\ &\leq 2\epsilon \sup_{t\in(0,1)} \|f_t(T)\| + \epsilon\end{aligned}$$

for r, s close to 1. Note that $\|f_t(T)\| \leq \|f_t\|_{\infty,\mathbb{D}}$. Indeed, the assumption 1. implies by theorem 4.4 that $\|p(T)\| = \|p\circ\tau(A)\| \leq \|p\circ\tau\|_{\infty,\frac{\pi}{2}} = \|p\|_{\infty,\mathbb{D}}$ for any polynomial p. By means of the density of the polynomials in $A(\mathbb{D})$, we conclude that $\|f_t(T)\| \leq \|f_t\|_{\infty,\mathbb{D}} \leq \|f\|_{\infty,\mathbb{D}}$ and finally get (5.2). Now define

$$\text{for } f \in H^\infty(\mathbb{D}) : \ \Psi(f)x = \lim_{r\to 1} f_r(T)x.$$

Then $\Psi(p) = p(T)$ for any polynomial p, since $\|p_r(T) - p(T)\| \leq \|p_r - p\|_{\infty,\mathbb{D}} \to 0$ for $r \to 1$. It is clear that $\Psi : H^\infty(\mathbb{D}) \to B(X)$ is linear and contractive. To show the multiplicativity, let $f, g \in H^\infty(\mathbb{D})$. Then $\Psi(fg)x = \lim_r (fg)_r(T)x = \lim_r f_r(T)g_r(T)x = \lim_r f_r(T)\,(\lim_s g_s(T)x) = \Psi(f)\Psi(g)x$, where the penultimate equality follows from $\sup_{r\in(0,1)}\|f_r(T)\| < \infty$. Now we pull back Ψ to $H^\infty(\Sigma_{\frac{\pi}{2}})$ and put

$$\Phi : H^\infty(\Sigma_{\frac{\pi}{2}}) \to B(X),\ \Phi(f) = \Psi(f\circ\zeta),$$

with $\zeta(z) = -\frac{z+1}{z-1}$ as in section 3. For a polynomial p, $\Phi(p\circ\tau) = \Psi(p) = p(T) = p\circ\tau(A)$, so that Φ coincides with Φ from theorem 4.4 on the set $\{p\circ\tau : p \text{ polynomial}\}$, which is dense in $A(\Sigma_{\frac{\pi}{2}})$. Therefore, Φ coincides on $A(\Sigma_{\frac{\pi}{2}})$ with the Φ from theorem 4.4.

We now show the convergence property of the calculus. Let $f, f_1, f_2, \ldots$ be as in the assumption. Put $g = f\circ\zeta$ and $g_n = f_n\circ\zeta$ and denote a_k and $a_k^{(n)}$ their Fourier coefficients. Then by Lebesgue's theorem, $g_n \to g$ in $L^2(\partial\,\mathbb{D})$. If $x \in (\frac{1}{2}+\frac{1}{2}A)^{-1}(D)$ and $x' \in X'$,

$$\begin{aligned}|\langle\Phi(f_n - f)x, x'\rangle| &= |\langle\Psi(g_n - g)x, x'\rangle| = |\lim_r \sum (a_k^{(n)} - a_k) r^k \langle T^k x, x'\rangle| \\ &= |\sum (a_k^{(n)} - a_k)\langle T^k x, x'\rangle| \leq \|(a_k^{(n)} - a_k)_k\|_{l^2} C_x \|x'\| \to 0.\end{aligned}$$

The statement for arbitrary $x \in X$ follows again from the density of $(\frac{1}{2}+\frac{1}{2}A)^{-1}(D)$ and $\|\Psi(g_n)\| \leq \|g_n\|_\infty \leq C$.

The uniqueness of the constructed functional calculus follows from this convergence property and another appeal to (4.1). □

The proof shows of course the following counterpart for contractions:

Corollary 5.3. *Let $T \in B(X)$ have a contractive $A(\mathbb{D})$-calculus and satisfy*

$$\sum_{k=0}^\infty |\langle T^k x, x'\rangle|^2 \leq C_x \|x'\|^2 \text{ for } x' \in X' \text{ and } x \text{ in a dense subset of } X.$$

Then $f(T)x = \lim_{r\to 1} f_r(T)x$ *extends the* $A(\mathbb{D})$*-calculus to a contractive algebra homomorphism* $H^\infty(\mathbb{D}) \to B(X)$ *which has the obvious analogue of the convergence property in theorem* 5.2*.*

Remark 5.4. To define an H^∞-calculus for a sectorial operator A on all of $H^\infty(\Sigma_{\frac{\pi}{2}})$, one usually needs the assumption $\overline{R(A)} = X$. In theorem 5.2, no such assumption is made explicitly. Nevertheless, any A satisfying (4.1) and (5.1) has dense image. Indeed, let $f_n(\lambda) = \frac{n\lambda}{1+n\lambda}$ and $f(\lambda) \equiv 1$. Then $f_n(\lambda) \to f(\lambda)$ for $\lambda \in i\mathbb{R}\backslash\{0\}$, and therefore, by theorem 5.2, $\Phi(f_n)x \to \Phi(f)x$ for every $x \in X$. Since $f, f_n \in R(\Sigma_\theta)$ for any $\theta \in (\frac{\pi}{2}, \pi)$, we know from theorem 4.4 that $\Phi(f), \Phi(f_n)$ are given by the H^∞-calculus. Thus, $\Phi(f_n)x = -AR(-\frac{1}{n}, A)x \in R(A)$ and $\Phi(f)x = x$. This shows $\overline{R(A)} = X$.

6. An example

The proof of theorem 4.2 shows that a θ-sectorial operator A on a Banach space X with $\theta < \frac{\pi}{2}$ has a contractive $H^\infty(\Sigma_{\frac{\pi}{2}})$-calculus if and only if for all Blaschke factors $b_w(\lambda) = (\lambda - w)(\lambda + \overline{w})^{-1}$ with $\operatorname{Re} w > 0$, we have $\|b_w(A)\| \le 1$, or, equivalently,

$$\| \operatorname{Id}_X + 2 \operatorname{Re} w R(-\overline{w}, A)\| \le 1 \text{ for all } \operatorname{Re} w > 0. \tag{6.1}$$

This condition allows one to show that many operators which are accretive in a scale of $L^p(K,\mu)$-spaces cannot have a contractive H^∞-calculus on the whole scale. Indeed if K is a compact metric space, we say that $T \in B(C(K))$ satisfies the Daugavet equation (see for example [WW]), if

$$\| \operatorname{Id}_X + T\| = 1 + \|T\|. \tag{6.2}$$

Hence if $X = C(K)$ and $T = -2\operatorname{Re}\lambda R(\lambda, A)$ satisfies (6.2) for some λ with $\operatorname{Re}\lambda < 0$, then (6.1) cannot hold. This is the case for a large class of operators A, for example when $R(\lambda, A)$ is weakly compact or an integral operator

$$Tf(y) = \int_K k(x,y)f(y)dy$$

with a measurable kernel. For these and weaker conditions for (6.2), see [WW].

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