APPLICATION OF THE MEAN ERGODIC THEOREM TO CERTAIN SEMIGROUPS

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Abstract. We study the asymptotic behaviour of solutions of the Cauchy problem
\[ u'(t) = (\sum_{j=1}^{n} (A_j + A_j^{-1}) - 2nI)u, \quad u(0) = x \text{ as } t \to \infty, \]
for invertible isometries \( A_1, \ldots, A_n \).

1. Introduction

Let \( E \) be a complex Banach space, \( L(E) \) the Banach algebra of all bounded linear operators on \( E \), and let \( A_1, \ldots, A_n \in L(E) \) be invertible, pairwise commuting, and such that \( \|A_k\| = \|A_k^{-1}\| = 1 \) (\( k = 1, \ldots, n \)). Let \( T_1, \ldots, T_n \in L(E) \) be defined by \( T_k = A_k + A_k^{-1} - 2I \), and let \( T = T_1 + \cdots + T_n \). The aim of this paper is to clear the asymptotic behaviour of the Cauchy problem
\[ u'(t) = Tu(t), \quad u(0) = u_0 \]
that is of \( t \to \exp(tT)u_0 \) for \( t \to \infty \). Such problems occur in a natural way by semidiscretization of the parabolic Cauchy problem \( v_t = \Delta v, \quad v(0, x) = v_0(x) \): For example, if \( v_0 : \mathbb{R}^n \to \mathbb{R} \) is bounded, the longitudinal line method, see for example [4], with step size 1 leads to a linear Cauchy problem of type (1.1) in \( l^\infty(\mathbb{Z}^n) \) with
\[ A_k x = (x(j_1, j_2, \ldots, j_{k-1}, j_k + 1, j_{k+1}, \ldots, j_n))_{j \in \mathbb{Z}^n}. \]
The corresponding problem for the heat equation was studied in [1].

2. Notations and preliminaries

For \( A \in L(E) \) let \( N(A) \), \( A(E) \), \( \sigma(A) \) and \( r(A) \) denote the kernel, the range, the spectrum and the spectral radius of \( A \), respectively. Let \( \mathbb{D} \) denote the complex unit circle \( \{ z \in \mathbb{C} : |z| < 1 \} \).

Proposition 2.1. Let \( A \in L(E) \), \( 0 \notin \sigma(A) \) and \( \|A\| = \|A^{-1}\| = 1 \). Then:
(1) \( A \) is an isometry;

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(2) \( \|A^n\| = \|A\|^n = 1 \) (\( n \in \mathbb{N} \)), hence \( A \) is normaloid;
(3) \( r(A) = 1 \) and \( \sigma(A) \subseteq \partial \mathbb{D} \);
(4) \( N(A - I) \cap (A - I)(E) = \{0\} \);
(5) \( (A - I)(E) = (A^{-1} - I)(E) \);
(6) \( N(A - I) = N((A - I)^2) \);
(7) \( N(A - I) \oplus (A - I)(E) \) is closed;
(8) if \( (A - I)(E) \) is closed then \( E = N(A - I) \oplus (A - I)(E) \).

**Proof.** (1) and (2) are obvious.

(3): From (2) we get \( r(A) = 1 \). Next, it is clear that

\[ \sigma(A) \cup \sigma(A^{-1}) \subseteq \overline{\mathbb{D}}. \]

Since \( \sigma(A) = \{ z \in \mathbb{C} : z^{-1} \in \sigma(A^{-1}) \} \), we conclude \( \sigma(A) \subseteq \partial \mathbb{D} \).

(4): Let \( x \in N(A - I) \cap (A - I)(E) \), let \( \varepsilon > 0 \) and choose \( z \in E \) such that \( \|x - (A - I)z\| < \varepsilon \). According to [2, Satz 102.3], we have \( \|x\| \leq \|x - (A - I)z\| < \varepsilon \), hence \( x = 0 \).

(5): Follows from \( (A - I)x = (I - A^{-1})(Ax) \).

(6): Follows from [2, Satz 102.3].

(7): Choose a sequence \( (x_n) \) in \( N(A - I) \oplus (A - I)(E) \) with \( x_n \to x_0 \) and corresponding decompositions \( x_n = y_n + z_n \). According to [2, Satz 102.3] we have

\[ \|y_n - y_m\| \leq \|x_n - x_m\| \quad (n, m \in \mathbb{N}), \]

hence \( (y_n) \) is convergent to a vector \( y_0 \in N(A - I) \). Thus

\[ z_n = x_n - y_n \to x_0 - y_0 \in (A - I)(E), \]

and therefore \( x_0 \in N(A - I) \oplus (A - I)(E) \).

(8): Follows from [2, Satz 72.4 and 102.4]. \( \square \)

**Proposition 2.2.** Let \( A \in L(E) \) be as in Proposition 2.1, let \( T = A + A^{-1} - 2I \), and let \( c : [0, \infty) \to \mathbb{R} \) denote the function

\[ c(t) = \exp(-t) \left( 1 + \sum_{n=0}^{\infty} \frac{tn^n}{n!} \right) \left( 1 - \frac{t}{n + 1} \right). \]

We have

(1) \( \|\exp(Tt)\| \leq 1 \) (\( t \geq 0 \));
(2) \( t \mapsto \sqrt{t}c(t) \) is bounded on \([0, \infty)\) and

\[ \|\exp(Tt)(A - I)x\| \leq c(t)\|x\| \quad (t \geq 0, \ x \in E); \]
(3) \( \lim_{t \to \infty} \exp(Tt)y = 0 \) (\( y \in (A - I)(E) \));
(4) if \( y \in E \) then

\[ \lim_{t \to \infty} \exp(Tt)y = 0 \iff y \in (A - I)(E); \]
(5) \( N(A - I) = \{ x \in E : \exp(Tt)x = x \ (t \geq 0) \} \).
Proof. (1): For each $t \geq 0$

\[
\| \exp(tT) \| = \| \exp(-2t) \exp(tA) \exp(tA^{-1}) \| \\
\leq \exp(-2t) \exp(t\|A\|) \exp(t\|A^{-1}\|) = 1.
\]

(2): Since $T = (A^{-1} - I) + (A - I)$ we have

\[
\exp(tT)(A - I) = \exp(t(A^{-1} - I)) \exp(t(A - I))(A - I) \quad (t \in \mathbb{R}),
\]

and

\[
\exp(t(A - I))(A - I)x = \exp(-t) \sum_{n=0}^{\infty} \frac{t^n}{n!} (A^{n+1} - A^n)x
\]

\[
= \exp(-t) \left( \sum_{n=0}^{\infty} \frac{t^n}{n!} \left( 1 - \frac{t}{n+1} \right) \right) A^{n+1}x - \exp(-t)x.
\]

Hence, since $\|A\| = 1$,

\[
\| \exp(tT)(A - I)x \| \leq \| \exp(t(A^{-1} - I)) \| \| \exp(t(A - I))(A - I)x \|
\]

\[
\leq c(t)\|x\| \quad (t \geq 0, \ x \in E).
\]

To see that $t \mapsto \sqrt{tc(t)}$ is bounded on $[0, \infty)$ let $N \in \mathbb{N}$ and $N \leq t \leq N + 1$. Then

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} \left| 1 - \frac{t}{n+1} \right| = \sum_{n=0}^{N-1} \frac{t^n}{n!} \left( 1 - \frac{t}{n+1} \right) + \sum_{n=N}^{\infty} \frac{t^n}{n!} \left( 1 - \frac{t}{n+1} \right)
\]

\[
= 2 \frac{t^N}{N!} - 1,
\]

and therefore

\[
\sqrt{tc(t)} = \sqrt{t} \exp(-t) \left( 1 + \sum_{n=0}^{\infty} \frac{t^n}{n!} \left| 1 - \frac{t}{n+1} \right| \right)
\]

\[
\leq \sqrt{N + 1} \exp(-N) \left( 1 + 2 \frac{(N+1)^N}{N!} - 1 \right)
\]

\[
= 2 \frac{(N+1)^{N+1/2}}{N!} \exp(-N),
\]

which is bounded according to Stirling’s formula.

(3): Follows from (2).

(4): The implication $\Leftarrow$ follows from (3). Now suppose that $\exp(tT)y \to 0$ as $t \to \infty$. Since

\[
\exp(tT)y = y + \sum_{n=1}^{\infty} \frac{t^n}{n!} T^n y
\]

\[
= y + (A - I) \sum_{n=1}^{\infty} \frac{t^n}{n!} (A - I)^{n-1} (I - A^{-1})^n y
\]

we conclude $y \in (A - I)(E)$.

(5): The inclusion

\[
N(A - I) \subseteq \{ x \in E : \exp(tT)x = x \quad (t \geq 0) \}
\]
is obvious. Now suppose that \( x \in E \) and \( \exp(tT)x = x \ (t \geq 0) \). By differentiation
\( 0 = T \exp(tT)x \ (t \geq 0) \), thus \( A^{-1}(A - I)^2x = Tx = 0 \). Part (6) of Proposition 2.1
now shows that \( x \in N(A - I) \).

\[
\begin{align*}
\text{3. The asymptotic behaviour of } \exp(tT) \\
\text{Theorem 3.1. Let } A \text{ and } T \text{ be as in Proposition 2.1. For } x \in E \text{ the following}
\text{assertions are equivalent:} \\
(1) \lim_{t \to \infty} \exp(tT)x \text{ exists in } E \ [\text{resp. } \lim_{t \to \infty} \exp(tT)x = 0]; \\
(2) x \in N(A - I) \oplus (A - I)(E) \ [\text{resp. } x \in (A - I)(E)]; \\
(3) \text{the sequence} \left( x + Ax + \cdots + A^mx \right)_{m \in \mathbb{N}} \\
\text{is convergent} \ [\text{resp. is convergent with limit } 0].
\end{align*}
\]

\text{Proof. That (2) implies (1) follows from Proposition 2.2. Now, assume that (1) holds, and let } \( z = \lim_{t \to \infty} \exp(tT)x \). \text{As in the proof of part (4) of Proposition 2.2}
\[ \exp(tT)x = x + (A - I) \sum_{n=1}^{\infty} \frac{t^n}{n!} (A - I)^{n-1}(I - A^{-1})^nx, \]
hence \( x - z \in (A - I)(E) \). [In particular, if \( z = 0 \) then \( x \in (A - I)(E) \).] From
part (3) of Proposition 2.2 we obtain
\[ (A - I)z = \lim_{t \to \infty} \exp(tT)(A - I)x = 0, \]
and therefore \( x = z + (x - z) \in N(A - I) \oplus (A - I)(E) \).
The equivalence of (2) and (3) is proved in [3, Ch.2, Theorem 1.3]. \( \square \)

According to part (8) of Proposition 2.1 the following corollary shows, that
\( \lim_{t \to \infty} \exp(tT)x \) exists for each \( x \in E \) if \( T(E) \) is closed:

\text{Corollary 3.1. We have}
\[ (1) T(E) = (A - I)^2(E) \subseteq (A - I)(E) \subseteq \overline{T(E)}; \]
\[ (2) T(E) = \overline{T(E)} \iff (A - I)^2(E) = (A - I)(E) \iff (A - I)(E) = \overline{(A - I)(E)}. \]

\text{Proof. (1): Part (5) of Proposition 2.1 gives}
\[ T(E) = (A - I)^2(E) \subseteq (A - I)(E). \]
As in the proof of Theorem 3.1 we obtain \( (A - I)(E) \subseteq \overline{T(E)} \). Now, (2) follows by
[2, Satz 102.4]. \( \square \)
4. The general case

Now, let $A_1, \ldots, A_n, T_1, \ldots, T_n$ and $T$ be as in section 1. Moreover we introduce the following subspaces of $E$:

$$X_1 = \bigcap_{j=1}^{n} N(A_j - I), \quad X_2 = \sum_{j=1}^{n} (A_j - I)(E), \quad X = X_1 + X_2.$$  

**Theorem 4.1.** Under the assumptions above

1. $X_2 = \{x \in E : \lim_{t \to \infty} \exp(tT)x = 0\}$;
2. $X_1 = \{x \in E : \exp(tT)x = x \ (t \geq 0)\}$;
3. $X = \{x \in E : \lim_{t \to \infty} \exp(tT)x \text{ exists in } E\}$;
4. $X_1 \cap X_2 = \{0\}$, and $X$ is closed.

**Proof.** (1): Let $x \in X_2$. Then $x = \lim_{m \to \infty} x_m$ where $x_m \in \sum_{j=1}^{n} (A_j - I)(E)$.

By part (1) and part (3) of Proposition 2.2 we obtain $\lim_{t \to \infty} \exp(tT)x_m = 0 \ (m \in \mathbb{N})$.

Let $\varepsilon > 0$, and choose $N \in \mathbb{N}$ such that $\|x - x_N\| < \varepsilon/2$. Next, choose $t_0 \in [0, \infty)$ such that $\|\exp(tT)x_N\| < \varepsilon/2 \ (t \geq t_0)$. Then

$$\|\exp(tT)x\| = \|\exp(tT)(x - x_N) + \exp(tT)x_N\| \leq \|x - x_N\| + \|\exp(tT)x_N\| < \varepsilon \ (t \geq t_0).$$

Thus $\lim_{t \to \infty} \exp(tT)x = 0$.

Now suppose that $x \in E$ and $\lim_{t \to \infty} \exp(tT)x = 0$. Set

$$h(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} T^n x.$$  

Since $T_j = (A_j - I)(I - A_j^{-1}) \ (j = 1, \ldots, n)$, we have

$$Tx = \sum_{j=1}^{n} (A_j - I)(I - A_j^{-1})x \in \sum_{j=1}^{n} (A_j - I)(E),$$  

hence

$$h(t) \in \sum_{j=1}^{n} (A_j - I)(E).$$

Thus, $\exp(tT)x = x + h(t)$ and $\lim_{t \to \infty} \exp(tT)x = 0$ imply $x \in X_2$.

(2): The inclusion $\subseteq$ is obvious. For the reversed inclusion let $x \in E$ be such that $\exp(tT)x = x \ (t \geq 0)$. Then by part (1) we obtain

$$x = (A_j - I)x = \exp(tT)(A_j - I)x \to 0 \ (t \to \infty) \ (j = 1, \ldots, n).$$
hence \( x \in X_1 \).

(3): Here, the inclusion \( \subseteq \) follows from parts (1) and (2) directly. Now, assume that \( x \in E \) is such that \( \lim_{t \to \infty} \exp(tT)x = z \). As in the proof of part (1)

\[
\exp(tT)x = x + h(t), \quad h(t) \in X_2.
\]

Therefore \( x - z \in X_2 \). From part (1) we derive

\[
(A_j - I)z = \lim_{t \to \infty} \exp(tT)(A_j - I)x = 0 \quad (j = 1, \ldots, n).
\]

Thus \( z \in X_1 \), and so \( x = z + (x - z) \in X_1 \oplus X_2 = X \).

(4): Let \( x \in X_1 \cap X_2 \). Then, by parts (1) and (2), we have

\[
\exp(tT)x = x \quad (t \geq 0), \quad \exp(tT)x \to 0 \quad (t \to \infty),
\]

thus \( x = 0 \). Next, if \((x_m)\) is a sequence in \( X \) with limit \( x_0 \), then there exist sequences \((y_m)\) and \((z_m)\) in \( X_1 \) and \( X_2 \), respectively, with \( x_m = y_m + z_m \). From part (1) and part (2) we obtain

\[
\exp(tT)(x_m - x_k) \to y_m - y_k \quad (t \to \infty).
\]

Since \( \|\exp(tT)(x_m - x_k)\| \leq \|x_m - x_k\| \quad (t \geq 0) \), we have

\[
\|y_m - y_k\| \leq \|x_m - x_k\|,
\]

thus \((y_m)\) is convergent. Let \( y_0 = \lim_{m \to \infty} y_m \). Then \( z_m = x_m - y_m \to x_0 - y_0 \). Hence we have \( y_0 \in X_1 \), \( x_0 - y_0 \in X_2 \), and therefore \( x_0 \in X_1 \oplus X_2 = X \). \( \square \)

The following result provides sufficient conditions for the convergence of \( \exp(tT)x \).

**Theorem 4.2.** Let \((k_1, \ldots, k_n) \in \mathbb{N}_0^n \), and set \( B = A_1^{k_1} \cdots A_n^{k_n} \). We have

1. \( \bigcap_{j=1}^n \left( N(A_j - I) \oplus (A_j - I)(E) \right) \subseteq X \);
2. \( (B - I)(E) \subseteq X_2 \);
3. if \( x \in E \) and if the sequences

\[
\left( \frac{x + A_jx + \cdots + A_j^mx}{m+1} \right)_{m \in \mathbb{N}}
\]

are convergent \((j = 1, \ldots, n)\), then \( \lim_{t \to \infty} \exp(tT)x \) exists in \( E \);
4. if \( x \in E \) and if the sequence

\[
\left( \frac{x + Bx + \cdots + B^mx}{m+1} \right)_{m \in \mathbb{N}}
\]

is convergent to 0 in \( E \), then \( \lim_{t \to \infty} \exp(tT)x = 0 \).

**Proof.** According to Theorem 3.1 we see that (3) follows from (1), and (4) follows from (2).

For the proof of (1) we use induction. If \( n = 1 \) the result follows by Theorem 3.1.
Suppose that \( n \in \mathbb{N} \) and that (1) holds. In the case of \( n + 1 \) operators \( T_1, \ldots, T_{n+1} \) we write \( T_0 = T_1 + \cdots + T_n \), so \( T = T_0 + T_{n+1} \). Let
\[
x \in \bigcap_{j=1}^{n+1} \left( N(A_j - I) \oplus (A_j - I)(E) \right).
\]
then
\[
x \in \bigcap_{j=1}^{n} \left( N(A_j - I) \oplus (A_j - I)(E) \right), \quad x \in N(A_{n+1} - I) \oplus (A_{n+1} - I)(E),
\]
and therefore the limits \( \lim_{t \to \infty} \exp(tT_0)x \) and \( \lim_{t \to \infty} \exp(tT_{n+1})x \) exist in \( E \).

From
\[
\| \exp(tT)x - \exp(sT)x \| = \| \exp(tT_0) \exp(tT_{n+1})x - \exp(sT_0) \exp(sT_{n+1})x \|
\]
\[
= \| \exp(tT_0) (\exp(tT_{n+1}) - \exp(sT_{n+1}))x + \exp(sT_{n+1}) (\exp(tT_0) - \exp(sT_0))x \|
\]
\[
\leq \| \exp(tT_{n+1})x - \exp(sT_{n+1})x \| + \| \exp(tT_0)x - \exp(sT_0)x \|
\]
we see that \( \lim_{t \to \infty} \exp(tT)x \) exists.

Next, we prove (2) for \( (k_1, \ldots, k_n) \in \mathbb{N}^n \), without loss of generality. Let \( p(z) = z_1^{k_1} \cdots z_n^{k_n} - 1 \ (z = (z_1, \ldots, z_n)) \), and note that there are polynomials \( q_1, \ldots, q_n \in \mathbb{C}[z_1, \ldots, z_n] \) such that
\[
p(z) = (z_1 - 1)q_1(z) + \cdots + (z_n - 1)q_n(z).
\]
Hence
\[
(B-I)x \in \sum_{j=1}^{n} (A_j - I)(E) \quad (x \in E),
\]
and therefore \( (B-I)(E) \subseteq X_2 \).

5. Example

Let us return to the semidiscretization of \( v_t = \Delta v \) in \( \mathbb{R}^2 \), that is we consider \( E = l^\infty(\mathbb{Z}^2) \) and
\[
A_1x = (x(i + 1, j))(i,j) \in \mathbb{Z}^2, \quad A_2x = (x(i, j + 1))(i,j) \in \mathbb{Z}^2.
\]
Let \( k_1, k_2 \in \mathbb{N} \), and assume that \( x \in l^\infty(\mathbb{Z}^2) \) is such that the sequence
\[
\left( \frac{x(i, j) + x(i + k_1, j + k_2) + \cdots + x(i + mk_1, j + mk_2)}{m + 1} \right)_{(i,j) \in \mathbb{Z}^2}
\]
tends to 0 as \( m \to \infty \) in \( l^\infty(\mathbb{Z}^2) \). Then
\[
\exp(tT)x \to 0 \quad (t \to \infty)
\]
(apply part (4) of Theorem 4.2 with \( B = A_1^{k_1} A_2^{k_2} \)).
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