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ABSTRACT

Let A be a unital Banach algebra and K an inessential ideal of A. We investigate the spectral properties of a holomorphic function f (defined on a region in $\mathbf{C}$) where the values of this function are K-Atkinson elements of A (i.e. each $f(\lambda)$ is left or right invertible modulo K).

Introduction

Let X denote a complex Banach space, $\mathscr{L}(X)$ the set of all bounded linear operators on X, and $\Phi(X)$ the set of all Fredholm operators in $\mathscr{L}(X)$. In [7], Gramsch proved the following theorem:

let G be a region in $\mathbf{C}$ and $T : G \to \mathscr{L}(X)$ a holomorphic operator function such that $T(\lambda) \in \Phi(X)$ for all $\lambda \in G$. Then there exist a discrete subset M of G and constants n, $m \geqslant 0$ with the following properties:

$$\dim N(T(\lambda)) = n \quad \textit{and} \quad \operatorname{codim} T(\lambda)(X) = m \textit{ for } \lambda \in G \backslash M,$$
$$\dim N(T(\lambda)) > n \quad \textit{and} \quad \operatorname{codim} T(\lambda)(X) > m \textit{ for } \lambda \in M,$$
$$\operatorname{ind} T(\lambda) = n - m \textit{ for all } \lambda \in G.$$

($N(T(\lambda))$ denotes the kernel of $T(\lambda)$, $T(\lambda)(X)$ denotes the range of $T(\lambda)$.)

The aim of this paper is to extend the above result from an operator-valued function T to a holomorphic function f (defined on a region in $\mathbf{C}$) with values in a complex Banach algebra A. The values of this function f are assumed to be left or right invertible modulo K, where K denotes an inessential ideal of A.

In the first section we give the preliminary definitions and results which we need in the sequel. Sections 2 and 3 deal with the basic Atkinson and Fredholm theory in semisimple Banach algebras. General Banach algebras are considered in section 4. In section 5 we consider holomorphic functions with values in a complex Banach algebra. In particular, we extend some results due to Gramsch [7] and Rowell [10].

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1. Preliminaries and notations

In this paper we always assume that A is a complex Banach algebra with identity $e \neq 0$.

Given a left ideal L of A the *quotient* is the ideal $L : A = \{a \in A : aA \subseteq L\}$. The quotient of a maximal left ideal is called a *primitive ideal*. We denote the set of primitive ideals by $\Pi(A)$. Observe that each $P \in \Pi(A)$ is closed.

If $J \subseteq A$ is non-empty and $\Omega \subseteq \Pi(A)$, we define

$$h(J) = \{P \in \Pi(A) : J \subseteq P\} \text{ and } k(\Omega) = \bigcap_{P \in \Omega} P.$$

The *radical* of A is the intersection of the primitive ideals of A and is denoted by $\mathrm{rad}(A)$. A is said to be *semisimple* if $\mathrm{rad}(A) = \{0\}$. A is said to be *primitive* if $\{0\} \in \Pi(A)$ (a primitive Banach algebra is semisimple). Let $P \in \Pi(A)$, then A/P is primitive [5, prop. 26.9].

In a semisimple Banach algebra A, the *socle* of A, $\mathrm{soc}(A)$, is defined to be the sum of all minimal right ideals (which equals the sum of all minimal left ideals [5, prop. 30.10]) or $\{0\}$ if A has no minimal right ideals. *Thus* $\mathrm{soc}(A)$ *is an ideal of* A.

For each subset M of A the *left annihilator* and the *right annihilator* are the sets

$$L(M) = \{y \in A : yM = 0\} \text{ and } R(M) = \{y \in A : My = 0\} \text{ respectively.}$$

If $M = \{x\}$ we simply write $L(x)$ and $R(x)$. Since A has an identity, we have

$$L(xA) = L(x) \text{ and } R(Ax) = R(x).$$

Let X be a complex Banach space, and let $\mathscr{L}(X)$ be the Banach algebra of bounded linear operators on X. If $T \in \mathscr{L}(X)$, we denote by $N(T)$ its kernel and by $T(X)$ its range.

2. Atkinson and Fredholm theory in semisimple Banach algebras

Fredholm theory in semiprime rings was pioneered by Barnes [2], [3]. This theory was then extended by Schreieck [11] and Weckbach [12] to elements of a semiprime algebra A, which are left or right invertible modulo $\mathrm{soc}(A)$.

The main references concerning Atkinson and Fredholm theory are [2], [3], [10], [11], [12] and the monograph [4] of Barnes, Murphy, Smyth and West.

Throughout this section, A will denote a semisimple Banach algebra.

2.1 Definition. The *ideal of inessential elements* of A is given by $I(A) = k(h(\mathrm{soc}(A)))$. An ideal K of A is called *inessential* if $K \subseteq I(A)$.

2.2 Definition. Let K be an inessential ideal of A. An element $x \in A$ is called a *K-Atkinson element* of A if x is left or right invertible modulo K. To be more precise, we define:

$$\Phi_l(A, K) = \{x \in A\colon \text{there exists } y \in A \text{ with } yx - e \in K\};$$
$$\Phi_r(A, K) = \{x \in A\colon \text{there exists } y \in A \text{ with } xy - e \in K\}.$$

The set of K-Atkinson elements is

$$\mathscr{A}(A, K) = \Phi_l(A, K) \cup \Phi_r(A, K).$$

The set of *K-Fredholm elements* of A is defined to be

$$\Phi(A, K) = \Phi_l(A, K) \cap \Phi_r(A, K).$$

The following characterisation of Atkinson elements is due to Barnes [3, theorem 2.3] and Rowell [10, prop. 2.13, 2.19].

2.3 Proposition. (a) $\Phi_l(A, \operatorname{soc}(A)) = \Phi_l(A, I(A))$ *and* $\Phi_r(A, \operatorname{soc}(A)) = \Phi_r(A, I(A))$.

(b) *Let K be an inessential ideal of A and $x \in A$. Then $x \in \Phi_l(A, K)[\Phi_r(A, K)]$ if and only if there exists an idempotent $p \in \operatorname{soc}(A) \cap K$ such that $Ax = A(e-p)[xA = (e-p)A]$.*

PROOF. [4, F.1.10]; [10, prop. 2.13, 2.19]. ■

2.4 Proposition. *Let K be an inessential ideal of A.*

(a) $x, y \in \Phi_l(A, K)[\Phi_r(A, K)] \Rightarrow xy \in \Phi_l(A, K)[\Phi_r(A, K)]$.

(b) $x, y \in A$, $xy \in \Phi_l(A, K)[\Phi_r(A, K)] \Rightarrow y \in \Phi_l(A, K)[x \in \Phi_r(A, K)]$.

(c) $x \in \Phi_l(A, K)[\Phi_r(A, K)]$, $u \in K \Rightarrow x + u \in \Phi_l(A, K)[\Phi_r(A, K)]$.

PROOF. Straightforward. ■

We close this section with a proposition due to Schreieck [11, Satz 5.4]. First we need the following definition. Let $x \in A$. We say that x is *relatively regular* if there exists $y \in A$ such that $xyx = x$.

2.5 Proposition. *Let K be an inessential ideal of A. Then $x \in \Phi_l(A, K)[\Phi_r(A, K)] \Leftrightarrow x$ is relatively regular and $R(x) \subseteq K[L(x) \subseteq K]$.*

PROOF. ($\Rightarrow$) By Proposition 2.3(b) there exists $p = p^2 \in \operatorname{soc}(A) \cap K$ such that $Ax = A(e-p)$. Therefore $yx = e - p$ for some $y \in A$. Further, we have $R(x) = R(Ax) = pA$, thus $xp = 0$. It follows that $xyx = x - xp = x$ and $pA \subseteq K$.

($\Leftarrow$) Take $y \in A$ such that $xyx = x$. Put $p = e - yx$. It follows that $p^2 = p$, $Ax = Ayx$ and $R(x) = R(Ax) = R(Ayx) = R(A(e-p)) = pA$. Since $R(x) \subseteq K$, we have $p = e - yx \in K$. Thus $x \in \Phi_l(A, K)$. ■

3. Atkinson and Fredholm theory in primitive Banach algebras

In this section, A will be a primitive Banach algebra. A non-zero idempotent $e_0 \in A$ is called *mimimal* if e_0Ae_0 is a division algebra. Min(A) denotes the set of all minimal idempotents of A.

Note that $\text{soc}(A) \neq \{0\}$ if and only if $\text{Min}(A) \neq \emptyset$ [4, BA.3.1]. To avoid trivialities, we assume that $\text{Min}(A)$ is non-empty.

Fix $e_0 \in \text{Min}(A)$, and let

$$x \to \hat{x}: A \to \mathscr{L}(Ae_0)$$

denote the left regular representation of A on the Banach space Ae_0, that is $\hat{x}(y) = xy (y \in Ae_0)$. For details see [4, p. 30] or [9, corollary 2.4.16].

Note that

$$\hat{x}(Ae_0) = xAe_0 \text{ and } N(\hat{x}) = R(x) \cap Ae_0 = R(x)e_0.$$

It follows from [4, F.2.1] that $\dim \hat{x}(Ae_0)$, $\dim N(\hat{x})$ and $\text{codim } \hat{x}(Ae_0)$ ($=\dim (Ae_0/xAe_0)$) are independent of the particular choice of $e_0 \in \text{Min}(A)$.

3.1 Definition. For $x \in A$ we define the *rank* of x by $\text{rank}(x) = \dim \hat{x}(Ae_0)$ ($=\dim xAe_0$). The *nullity* of x is defined to be $\text{nul}(x) = \dim N(\hat{x})$. The *defect* of x is defined by $\text{def}(x) = \dim(Ae_0/xAe_0)$.

3.2 *Remark.* (a) If $Ax = A(e-p)$ and $p = p^2$, then

$$\begin{aligned} &R(x) = pA \text{ and} \\ &\text{nul}(x) = \dim R(x)e_0 = \dim pAe_0 = \text{rank}(p). \end{aligned} \tag{3.1}$$

(b) If $xA = (e-q)A$ and $q = q^2$, then

$$\begin{aligned} &Ae_0 = (e-q)Ae_0 \oplus qAe_0 = xAe_0 \oplus qAe_0 \text{ and} \\ &\text{def}(x) = \dim qAe_0 = \text{rank}(q). \end{aligned} \tag{3.2}$$

3.3 Theorem. (a) $x = 0 \Leftrightarrow \text{rank}(x) = 0$.
(b) $\text{soc}(A) = \{x \in A: \text{rank}(x) < \infty\}$.

The proof may be found in [4, F.2.4].

The next theorem is a characterisation of Atkinson elements in terms of nullity and defect.

3.4 Theorem [12, Satz 3.5]. $x \in \Phi_l(A, I(A))[\Phi_r(A, I(A))] \Leftrightarrow x$ *is relatively regular and* $\text{nul}(x) < \infty[\text{def}(x) < \infty]$.

Proof. 1. If $x \in \Phi_l(A, I(A))$ there exists $p = p^2 \in \text{soc}(A)$ such that $Ax = A(e-p)$ (Proposition 2.3). By Proposition 2.5 and Remark 3.2, we conclude that x is relatively regular and that $\text{nul}(x) = \text{rank}(p)$. Because of Theorem 3.3(b) and $p \in \text{soc}(A)$, it follows that $\text{nul}(x) < \infty$.

2. Take $y \in A$ such that $xyx = x$. Put $p = e - yx$. It follows that $p^2 = p$, $Ax = Ayx$ and $R(x) = R(Ax) = R(Ayx) = pA$. Thus $\text{rank}(p) = \dim pAe_0 = \dim R(x)e_0 = \text{nul}(x) < \infty$. From Theorem 3.3(b) we derive $p = e - yx \in \text{soc}(A)$, hence $x \in \Phi_l(A, \text{soc}(A)) = \Phi_l(A, I(A))$.

A similar proof deals with the case of $x \in \Phi_r(A, I(A))$. ∎

Let K be an inessential ideal of A. Since $\Phi_l(A, K) \subseteq \Phi_l(A, I(A))$ and $\Phi_r(A, K) \subseteq \Phi_r(A, I(A))$, it follows from Theorem 3.4 that for a K-Atkinson element x at least one of the quantities $\mathrm{nul}(x)$, $\mathrm{def}(x)$ is finite. Thus we are in a position to define the index for an Atkinson element.

3.5 Definition. The *index* of $x \in \mathscr{A}(A, K)$ is defined by $\mathrm{ind}(x) = \mathrm{nul}(x) - \mathrm{def}(x)$.

3.6 Proposition. *Let K be an inessential ideal of A.*

(a) $x \in \Phi_l(A, K)[\Phi_r(A, K)]$, $u \in K \Rightarrow x + u \in \Phi_l(A, K)[\Phi_r(A, K)]$ *and* $\mathrm{ind}(x+u) = \mathrm{ind}(x)$.

(b) *$x \in A$ is left invertible if and only if $x \in \Phi_l(A, K)$ and* $\mathrm{nul}(x) = 0$.

(c) *$x \in A$ is right invertible if and only if $x \in \Phi_r(A, K)$ and* $\mathrm{def}(x) = 0$.

PROOF. (a) [10, lemma 3.2(1)].

(b) ($\Rightarrow$) If x is left invertible, then $x \in \Phi_l(A, K)$ and $R(x) = \{0\}$. Hence $\mathrm{nul}(x) = 0$.

($\Leftarrow$) By Proposition 2.3, there exists $p = p^2 \in \mathrm{soc}(A) \cap K$ such that $Ax = A(e-p)$. Using Remark 3.2(a) this gives $R(x) = pA$ and $\mathrm{nul}(x) = \mathrm{rank}(p) = 0$. Hence $p = 0$ and $Ax = A$.

(c) ($\Rightarrow$) If x is right invertible, then $x \in \Phi_r(A, K)$ and $xA = A$. Hence $xAe_0 = Ae_0$ where $e_0 \in \mathrm{Min}(A)$. Thus $\mathrm{def}(x) = 0$.

($\Leftarrow$) By Proposition 2.3, there exists $q = q^2 \in \mathrm{soc}(A) \cap K$ such that $xA = (e-q)A$. Using Remark 3.2(b) this gives $\mathrm{def}(x) = \mathrm{rank}(q) = 0$. Hence $q = 0$ and $xA = A$. ∎

3.7 Theorem [12, theorem 3.7]. *Let K be an inessential ideal of A.*

(a) *If $x, y \in \Phi_l(A, K)[\Phi_r(A, K)]$, then* $\mathrm{ind}(xy) = \mathrm{ind}(x) + \mathrm{ind}(y)$.

(b) *If $xy \in \Phi(A, K)$, then* $\mathrm{ind}(x) = \mathrm{ind}(xy) - \mathrm{ind}(y)$.

PROOF. (a) It suffices to consider only the case where $x, y \in \Phi_l(A, K)$.

Case 1: $x, y \in \Phi(A, I(A)) = \Phi(A, \mathrm{soc}(A))$. Using [4, theorem F.2.9] this gives $\mathrm{ind}(xy) = \mathrm{ind}(x) + \mathrm{ind}(y)$.

Case 2: $x \notin \Phi(A, \mathrm{soc}(A))$ or $y \notin \Phi(A, \mathrm{soc}(A))$. It follows from Proposition 2.4 that $xy \in \Phi_l(A, \mathrm{soc}(A)) \backslash \Phi_r(A, \mathrm{soc}(A))$. Hence $\mathrm{ind}(xy) = -\infty = \mathrm{ind}(x) + \mathrm{ind}(y)$.

(b) It follows from Proposition 2.4 that $x \in \Phi_r(A, \mathrm{soc}(A))$ and $y \in \Phi_l(A, \mathrm{soc}(A))$. If $x \in \Phi(A, \mathrm{soc}(A))$, then $y \in \Phi(A, \mathrm{soc}(A))$(Proposition 2.4). Now use (a). If $x \notin \Phi(A, \mathrm{soc}(A))$, then $x \notin \Phi_l(A, \mathrm{soc}(A))$ and $y \notin \Phi_r(A, \mathrm{soc}(A))$. Hence $\mathrm{ind}(x) = -\mathrm{ind}(y) = -\infty$. ∎

The next theorem shows that the sets

$$\Phi_l^{(n)}(A, K) := \{x \in \Phi_l(A, K) : \mathrm{ind}(x) = n\} (n \in \mathbf{Z} \cup \{-\infty\}),$$
$$\Phi_r^{(n)}(A, K) := \{x \in \Phi_r(A, K) : \mathrm{ind}(x) = n\} (n \in \mathbf{Z} \cup \{\infty\})$$

$$\text{and } \Phi^{(n)}(A, K) := \{x \in \Phi(A, K) : \text{ind}(x) = n\}(n \in \mathbf{Z})$$

are open subsets of A.

3.8 Theorem. *Let K be an inessential ideal of A. For each $x \in \mathscr{A}(A, K)$ there is a positive $\gamma(=\gamma(x))$ with the following properties: if $s \in A$ and $\|s\| < \gamma$, then*
(a) $x + s \in \mathscr{A}(A, K)$, $\text{ind}(x + s) = \text{ind}(x)$;
(b) $\text{nul}(x + s) \leqslant \text{nul}(x)$, $\text{def}(x + s) \leqslant \text{def}(x)$.

Proof. Let $x \in \Phi_l(A, K)$ (the proof for the case $x \in \Phi_r(A, K)$ is similar). By Proposition 2.3, we can find an idempotent $p \in \text{soc}(A) \cap K$ such that $Ax = A(e - p)$. Hence

$$yx = e - p \tag{3.3}$$

for some $y \in A$. Put $\gamma = \|y\|^{-1}$. Let $s \in A$ and $\|s\| < \gamma$, then $e + ys$ is invertible and

$$y(x + s) = e + ys - p. \tag{3.4}$$

Thus

$$(e + ys)^{-1}y(x + s) = e - (e + ys)^{-1}p, (e + ys)^{-1}p \in K, \tag{3.5}$$

which implies that $x + s \in \Phi_l(A, K)$.

From (3.3), (3.4) and Proposition 3.6 we derive yx, $y(x + s) \in \Phi(A, K)$ and $\text{ind}(yx) = \text{ind}(e - p) = \text{ind}(e) = 0 = \text{ind}(e + ys) = \text{ind}(e + ys - p) = \text{ind}(y(x + s))$.

Hence, by Theorem 3.7(b),

$$\text{ind}(x + s) = \text{ind}(y(x + s)) - \text{ind}(y) = \text{ind}(yx) - \text{ind}(y) = \text{ind}(x). \tag{3.6}$$

Next we show $\text{nul}(x + s) \leqslant \text{nul}(x)$. Let $a \in R(x + s)$, then $0 = (e + ys)^{-1}y(x + s)a = a - (e + ys)^{-1}pa$ and thus $a \in (e + ys)^{-1}pA$. Hence $R(x + s) \subseteq (e + ys)^{-1}pA$ and

$$R(x + s)e_0 \subseteq (e + ys)^{-1}pAe_0 \,(e_0 \in \text{Min}(A)).$$

This shows $\text{nul}(x + s) \leqslant \text{rank}(p) = \text{nul}(x)$ (Remark 3.2(a)). In view of (3.6), we conclude that $\text{def}(x + s) \leqslant \text{def}(x)$. ■

Now we consider the special Banach algebra $\mathscr{L}(X)$ where X is a complex Banach space. For this purpose we need the following two classes of bounded linear operators:

$\mathscr{F}(X)$ the ideal of finite rank operators in $\mathscr{L}(X)$;

$\mathcal{K}(X)$ the closed ideal of compact operators on X.

3.9 *Example*. (a) $\mathcal{L}(X)$ is primitive.

(b) $\mathrm{soc}(\mathcal{L}(X)) = \mathcal{F}(X)$, $\mathrm{Min}(\mathcal{L}(X)) = \{P \in \mathcal{L}(X)\colon P^2 = P \text{ and } \dim P(X) = 1\}$.

(c) For $T \in \mathcal{L}(X)$ we have $\mathrm{nul}(T) = \dim N(T)$ and $\mathrm{def}(T) = \mathrm{codim}\, T(X)$.

(d) $\mathcal{K}(X)$ is an inessential ideal of $\mathcal{L}(X)$.

(e) An operator T in $\mathcal{L}(X)$ is relatively regular with $\mathrm{nul}(T) < \infty$ or $\mathrm{def}(T) < \infty$ if and only if $T \in \mathcal{A}(\mathcal{L}(X), \mathcal{K}(X))$.

PROOF. (a), (b), (c) [4, F.2.2.].

(d) [8, Satz 106.2].

(e) [6, p. 28]. ■

An operator $T \in \mathcal{A}(\mathcal{L}(X), \mathcal{K}(X))$ is called an *Atkinson operator*.

Using Theorem 3.4 and the definition of nullity and defect, the following result is easy to confirm.

3.10 Proposition. *Let K be an inessential ideal and $e_0 \in \mathrm{Min}(A)$. If $x \in \mathcal{A}(A, K)$, then $\hat{x}$ is an Atkinson operator on Ae_0.*

Let X^* denote the conjugate space of the Banach space X. The adjoint of a linear operator T in $\mathcal{L}(X)$ is denoted by T^*.

The next proposition will be needed in section 5.

3.11 Proposition. *If $T \in \mathcal{L}(X)$ is an Atkinson operator, then T^* is an Atkinson operator and*

$$\mathrm{nul}(T) = \mathrm{def}(T^*) \text{ and } \mathrm{def}(T) = \mathrm{nul}(T^*).$$

PROOF. Clearly, T^* is relatively regular. Using [8, Satz 82.1], the result follows. ■

4. General Banach algebras

In this section we assume that A is an arbitrary Banach algebra. Thus $\mathrm{soc}(A)$ might not exist. The quotient algebra $A' = A/\mathrm{rad}(A)$ is semisimple [5, prop. 24.21], hence A' has a socle.

We write x' for the coset $x + \mathrm{rad}(A) (x \in A)$ and if $S \subseteq A$ write $S' = \{x' : x \in S\}$.

4.1 Definition. (a) The *presocle* of A is defined by $\mathrm{psoc}(A) = \{x \in A\colon x' \in \mathrm{soc}(A')\}$.

(b) The *ideal of inessential elements* is defined to be $I(A) = k(h(\mathrm{psoc}(A)))$.

(c) An ideal K of A is *inessential* if $K \subseteq I(A)$.

Observe that $\mathrm{psoc}(A)$ is an ideal of A and that $\mathrm{soc}(A) = \mathrm{psoc}(A)$ if A is semisimple.

If K is an inessential ideal of A, the sets

$$\Phi_l(A, K),\ \Phi_r(A, K),\ \mathscr{A}(A, K) \text{ and } \Phi(A, K)$$

are defined as in Definition 2.2.

Notation. If $K = I(A)$ we write $\Phi_l(A)$, $\Phi_r(A)$, $\mathscr{A}(A)$, $\Phi(A)$ instead of $\Phi_l(A, I(A))$, $\Phi_r(A, I(A))$, $\mathscr{A}(A, I(A))$, $\Phi(A, I(A))$.

Recall that the quotient algebra A/P is primitive $(P \in \Pi(A))$.

4.2 Theorem. (a) $\Phi_l(A) = \Phi_l(A, \text{psoc}(A))$, $\Phi_r(A) = \Phi_r(A, \text{psoc}(A))$.

(b) *If* $x \in \Phi_l(A)[\Phi_r(A)]$ *there exist* $\epsilon > 0$ *and a finite subset* Ω *of* $\Pi(A)$ *such that if* $y \in A$ *and* $\|x - y\| < \epsilon$ *then*

(b.1) $y + P \in \Phi_l(A/P)[\Phi_r(A/P)]$ *for all* $P \in \Omega$,

(b.2) $y + P$ *is left* [*right*] *invertible for all* $P \in \Pi(A)\backslash\Omega$.

Proof. [10, prop. 2.19, theorem 2.22]. ■

4.3 Corollary. *If* $x \in \Phi_l(A)[\Phi_r(A)]$ *there exist* $P_1, \ldots, P_n \in \Pi(A)$ *such that*

$$x + P \in \Phi_l(A/P)[\Phi_r(A/P)] \textit{ for all } P \in \Pi(A) \textit{ and}$$

$$\text{nul}(x + P) = 0[\text{def}(x + P) = 0] \textit{ for all } P \in \Pi(A)\backslash\{P_1, \ldots, P_n\}.$$

Proof. Theorem 4.2; Proposition 3.6. ■

In view of Corollary 4.3 the concepts of nullity, defect and index can be extended as follows.

4.4 Definition. (a) If $x \in \mathscr{A}(A)$ the *nullity*, *defect* and *index functions* $\Pi(A) \to \mathbf{Z} \cup \{-\infty, \infty\}$ are defined by

$$\nu(x)(P) = \text{nul}(x + P),\ \delta(x)(P) = \text{def}(x + P),$$
$$\iota(x)(P) = \text{ind}(x + P).$$

(b) If $x \in \mathscr{A}(A)$ we define

$$\text{nul}(x) = \begin{cases} \sum_{P \in \Pi(A)} \nu(x)(P) & \text{for } x \in \Phi_l(A), \\ \infty & \text{for } x \notin \Phi_l(A) \end{cases}$$

$$\text{def}(x) = \begin{cases} \sum_{P \in \Pi(A)} \delta(x)(P) & \text{for } x \in \Phi_r(A), \\ \infty & \text{for } x \notin \Phi_r(A) \end{cases}$$

and $\text{ind}(x) = \text{nul}(x) - \text{def}(x)$.

Note that $\text{ind}(x) = \sum_{P \in \Pi(A)} \iota(x)(P)$ if $x \in \Phi(A)$.

4.5 *Remark.* If A is a primitive Banach algebra and $\{0\} \neq P \in \Pi(A)$ then $\mathrm{soc}(A) \subseteq P$ [4, p. 38]. Suppose $x \in \Phi_l(A)$. By Proposition 2.3 there are $y \in A$ and $p \in \mathrm{soc}(A)$ such that $yx = e - p$. It follows that $p \in P$ for all $P \in \Pi(A)$, $P \neq \{0\}$. Thus $x + P$ is left invertible in A/P for all $P \in \Pi(A)$, $P \neq \{0\}$. Proposition 3.6(b) gives $\nu(x)(P) = 0$ for all $P \in \Pi(A)$, $P \neq \{0\}$. Hence $\mathrm{nul}(x) = \nu(x)(\{0\})$.

Similar: $x \in \Phi_r(A) \Rightarrow \mathrm{def}(x) = \delta(x)(\{0\})$.

Now Proposition 3.6, Theorem 3.7 and Theorem 3.8 extend to the general case.

4.6 Proposition. *Let $x \in A$. Then x is left [right] invertible if and only if $x \in \Phi_l(A)[\Phi_r(A)]$ and $\nu(x)(P) = 0[\delta(x)(P) = 0]$ for all $P \in \Pi(A)$.*

PROOF. [10, prop. 2.18, 3.4]; Proposition 3.6. ■

4.7 Theorem (Index). *Let K be an inessential ideal of A.*

(a) *If $x,y \in \Phi_l(A, K)[\Phi_r(A, K)]$, then*

$$\iota(xy) \equiv \iota(x) + \iota(y) \quad \textit{and} \quad \mathrm{ind}(xy) = \mathrm{ind}(x) + \mathrm{ind}(y).$$

(b) *If $xy \in \Phi(A, K)$, then*

$$\iota(x) \equiv \iota(xy) - \iota(y) \quad \textit{and} \quad \mathrm{ind}(x) = \mathrm{ind}(xy) - \mathrm{ind}(x).$$

PROOF. The argument is analogous to the one in Theorem 3.7, with use being made of [4, F.3.8]. ■

4.8 Theorem. *Let K be an inessential ideal of A. For each $x \in \Phi_l(A, K)[\Phi_r(A, K)]$ there is a positive γ with the following properties: if $s \in A$ and $\|s\| < \gamma$, then*

(a) $x + s \in \Phi_l(A, K)[\Phi_r(A, K)]$;
(b) $\nu(x + s)(P) \leqslant \nu(x)(P)$ $[\delta(x + s)(P) \leqslant \delta(x)(P)]$ *for all* $P \in \Pi(A)$;
(c) $\mathrm{nul}(x + s) \leqslant \mathrm{nul}(x)$ $[\mathrm{def}(x + s) \leqslant \mathrm{def}(x)]$;
(d) $\iota(x + s) \equiv \iota(x)$;
(e) $\mathrm{ind}(x + s) = \mathrm{ind}(x)$.

PROOF. Let $x \in \Phi_l(A, K)$(the proof for the case $x \in \Phi_r(A, K)$ is similar). There exist $z \in A$ and $k \in K$ such that $zx = e - k$. By Theorem 4.2(a), we can find $y \in A$ and $p \in \mathrm{psoc}(A)$, such that

$$yx = e - p. \tag{4.1}$$

Put $\gamma_0 = \min\{\|y\|^{-1}, \|z\|^{-1}\}$. Let $s \in A$ and $\|s\| < \gamma_0$, then $e + ys$ and $e + zs$ are invertible, thus $(e + zs)^{-1}z(x + s) = e - (e + zs)^{-1}k$ and $(e+zs)^{-1}k \in K$. Hence $x + s \in \Phi_l(A, K)$. Since $yx = e - p$, we have

$$y(x + s) = (e + ys) - p, \tag{4.2}$$

therefore

$$yx, y(x+s) \in \Phi(A). \tag{4.3}$$

Since $p \in \text{psoc}(A)$, [4, BA.3.4] shows that $p'+P' \in \text{soc}(A'/P')(P \in \Pi(A))$, thus, by [4, BA.2.6],

$$p + P \in \text{soc}(A/P) \text{ for all } P \in \Pi(A). \tag{4.4}$$

Combining (4.3) and Corollary 4.3,

$$yx + P \in \Phi(A/P) \text{ for all } P \in \Pi(A). \tag{4.5}$$

In view of (4.4), (4.5) and Proposition 3.6(a), we conclude that

$$\iota(yx)(P) = \text{ind}(e - p + P) = \text{ind}(e + P) = 0 \tag{4.6}$$

for all $P \in \Pi(A)$.

So

$$\text{ind}(yx) = \sum_{P \in \Pi(A)} \iota(yx)(P) = 0. \tag{4.7}$$

Analogous arguments (use (4.2)) show that

$$\iota(y(x+s))(P) = \text{ind}(e + ys + P) = 0 \text{ for all } P \in \Pi(A) \tag{4.8}$$

and

$$\text{ind}(y(x+s)) = 0. \tag{4.9}$$

By Theorem 4.7(b), (4.8) and (4.9), we derive

$$\iota(x+s) \equiv \iota(x) \text{ and } \text{ind}(x+s) = \text{ind}(x).$$

So far, we have

$$\|s\| < \gamma_0 \Rightarrow x + s \in \Phi_l(A, K),$$
$$\iota(x+s) \equiv \iota(x) \text{ and } \text{ind}(x+s) = \text{ind}(x).$$

According to Theorem 4.2(b), there exist $\epsilon > 0$ and $P_1, \ldots, P_n \in \Pi(A)$ such that if $\|s\| < \epsilon$ then

$$x + s + P_j \in \Phi_l(A/P_j) \qquad (j = 1, \ldots, n) \tag{4.10}$$

and

$$x + s + P, x + P \text{ are left invertible for all } P \in \Pi(A) \backslash \{P_1, \ldots, P_n\}. \tag{4.11}$$

By Theorem 3.8, for each $j \in \{1, \ldots, n\}$, there exists $\gamma_j \in (0, \epsilon]$ such that if $\|s + P_j\| < \gamma_j$ then

$$\mathrm{nul}(x + s + P_j) \leqslant \mathrm{nul}(x + P_j). \tag{4.12}$$

Note that $\mathrm{nul}(x + s + P) = \mathrm{nul}(x + P) = 0$, whenever $P \in \Pi(A) \backslash \{P_1, \ldots, P_n\}$. Put $\gamma_{n+1} = \min\{\gamma_1, \ldots, \gamma_n\}$. From (4.12) we derive for $\|s\| < \gamma_{n+1}$

$$\nu(x + s)(P) \leqslant \nu(x)(P) \text{ for all } P \in \Pi(A).$$

Put $\gamma = \min\{\gamma_0, \gamma_{n+1}\}$ and the proof is complete. ■

5. Holomorphic functions

In this section G will denote a region in **C** and f a function on G with values in A.

As an immediate consequence of Theorem 4.8 we have the following proposition.

5.1 Proposition. *Let K be an inessential ideal of A. Suppose that f is continuous and $f(\lambda) \in \mathscr{A}(A, K)$ for all $\lambda \in G$. Then*

(a) $\mathrm{ind}(f(\lambda))$ *is constant on* G;

(b) *either* $\mathrm{nul}(f(\lambda)) = \infty$ *for all* $\lambda \in G$ *or* $\mathrm{nul}(f(\lambda)) < \infty$ *for all* $\lambda \in G$;

(c) *either* $\mathrm{def}(f(\lambda)) = \infty$ *for all* $\lambda \in G$ *or* $\mathrm{def}(f(\lambda)) < \infty$ *for all* $\lambda \in G$.

A subset M of the region G is called *discrete* if M has no accumulation points in G. Thus M is at most countable and $G \backslash M$ is again a region.

5.2 Lemma. *Let A be primitive and let f be holomorphic such that* $m = \max_{\lambda \in G} \mathrm{rank}\,(f(\lambda))$ *exists. Then there is a discrete subset M of G such that*

$$\mathrm{rank}(f(\lambda)) = m \text{ for all } \lambda \in G \backslash M.$$

PROOF. Fix $e_0 \in \mathrm{Min}(A)$, and let the operator-valued function $\hat{f}: G \to \mathscr{L}(Ae_0)$ be given by $\hat{f}(\lambda) = \widehat{f(\lambda)}$ for $\lambda \in G$. Since $\hat{f}$ is holomorphic and $\dim \hat{f}(\lambda)(Ae_0) = \dim \widehat{f(\lambda)}(Ae_0) = \mathrm{rank}(f(\lambda)) \leqslant m$ for all $\lambda \in G$, the result follows from [7, lemma 3.2]. ■

The idea of the next lemma goes back to a theorem of Gramsch [7, Satz 3.3].

5.3 Lemma. *Let X be a complete Banach space and $T: G \to \mathscr{L}(X)$ be a holomorphic function. If $T(\lambda) \in \mathscr{A}(\mathscr{L}(X), \mathscr{K}(X))$ for all $\lambda \in G$, then, for every $\lambda_0 \in G$, there exist a positive δ and constants α, $\beta \leqslant \infty$ such that*

$$\dim N(T(\lambda)) = \alpha \leqslant \dim N(T(\lambda_0)) \tag{5.1}$$

and

$$\operatorname{codim} T(\lambda)(X) = \beta \leqslant \operatorname{codim} T(\lambda_0)(X), \tag{5.2}$$

whenever $0 < |\lambda - \lambda_0| < \delta$.

PROOF. Take $\lambda_0 \in G$. Suppose first that $\dim N(T(\lambda_0)) < \infty$. In this case, the proof of (5.1) is contained in the proof of [7, Satz 3.3].

If $\dim N(T(\lambda_0)) = \infty$, then $\dim N(T(\lambda)) = \infty$ for all $\lambda \in G$ (Proposition 5.1). Thus (5.1) is proved.

Suppose now that $\operatorname{codim} T(\lambda_0)(X) < \infty$. Using Proposition 3.11 we have $T(\lambda)^* \in \mathscr{A}(\mathscr{L}(X^*), \mathscr{K}(X^*))$ and $\operatorname{codim} T(\lambda)(X) = \dim N(T(\lambda)^*)$ for all $\lambda \in G$. According to (5.1) there exist $\delta > 0$ and a constant β such that

$$\dim N(T(\lambda)^*) = \beta \leqslant \dim N(T(\lambda_0)^*) \quad (0 < |\lambda - \lambda_0| < \delta),$$

that is

$$\operatorname{codim} T(\lambda)(X) = \beta \leqslant \operatorname{codim} T(\lambda_0)(X),$$

whenever $0 < |\lambda - \lambda_0| < \delta$.

If $\operatorname{codim} T(\lambda_0)(X) = \infty$, then $\operatorname{codim} T(\lambda)(X) = \infty$ for all $\lambda \in G$ (Proposition 5.1). ■

The next theorem plays a central role in our investigations.

5.4 Theorem. *Let A be primitive, K an inessential ideal of A, and let $f: G \to A$ be holomorphic such that $f(\lambda) \in \mathscr{A}(A, K)$ for all $\lambda \in G$.*

(a) *If $f(\lambda) \in \Phi_l(A, K)$ for all $\lambda \in G$, then, for every $\lambda_0 \in G$, there exist a positive δ and a constant α such that*

$$\operatorname{nul}(f(\lambda)) = \alpha \leqslant \operatorname{nul}(f(\lambda_0)),$$

whenever $0 < |\lambda - \lambda_0| < \delta$.

(b) *If $f(\lambda) \in \Phi_r(A, K)$ for all $\lambda \in G$, then, for every $\lambda_0 \in G$, there exist a positive δ and a constant β such that*

$$\operatorname{def}(f(\lambda)) = \beta \leqslant \operatorname{def}(f(\lambda_0)),$$

whenever $0 < |\lambda - \lambda_0| < \delta$.

PROOF. Fix $e_0 \in \operatorname{Min}(A)$, and let the holomorphic operator-valued function: $\bar{f}: G \to \mathscr{L}(Ae_0)$ be given by $\bar{f}(\lambda) = \widehat{f(\lambda)}$ for $\lambda \in G$. It follows from Proposition 3.10 that $\bar{f}(\lambda)$ is an Atkinson operator on Ae_0, $\operatorname{nul}(f(\lambda)) = \dim N(\bar{f}(\lambda))$ and $\operatorname{def}(f(\lambda)) = \operatorname{codim} \bar{f}(\lambda)(Ae_0) (\lambda \in G)$. The result follows by Lemma 5.3. ■

Notation. For $\delta > 0$ and $\lambda_0 \in \mathbf{C}$ define

$$K_\delta(\lambda_0) = \{\lambda \in \mathbf{C} : |\lambda - \lambda_0| < \delta\} \text{ and } \dot{K}_\delta(\lambda_0) = K_\delta(\lambda_0) \backslash \{\lambda_0\}.$$

5.5 Theorem. *Let A be an arbitrary Banach algebra, K an inessential ideal of A, and let $f : G \to A$ be holomorphic.*

(a) *Suppose $\lambda_0 \in G$ and $f(\lambda) \in \Phi_l(A, K)$ for all $\lambda \in G$. Then there exists $\delta > 0$ such that $\nu(f(\lambda))$ is independent of λ for $0 < |\lambda - \lambda_0| < \delta$ and is bounded above by $\nu(f(\lambda_0))$.*

(b) *Suppose $\lambda_0 \in G$ and $f(\lambda) \in \Phi_r(A, K)$ for all $\lambda \in G$. Then there exists $\delta > 0$ such that $\delta(f(\lambda))$ is independent of λ for $0 < |\lambda - \lambda_0| < \delta$ and is bounded above by $\delta(f(\lambda_0))$.*

PROOF. (a) According to Theorem 4.2(b), there are $\epsilon > 0$ and $P_1, \ldots, P_n \in \Pi(A)$ such that if $\|f(\lambda) - f(\lambda_0)\| < \epsilon$ then

$$f(\lambda) + P_j \in \Phi_l(A/P_j) \qquad (j = 1, \ldots, n), \tag{5.3}$$

$$f(\lambda) + P, f(\lambda_0) + P \text{ are left invertible in } A/P \text{ for all } P \in \Pi(A)\backslash\{P_1, \ldots, P_n\}. \tag{5.4}$$

Choose now $\delta_0 > 0$ so that $\|f(\lambda) - f(\lambda_0)\| < \epsilon$ for $|\lambda - \lambda_0| < \delta_0$. For $P \in \Pi(A)$ let the holomorphic function $f_P : K_{\delta_0}(\lambda_0) \to A/P$ be given by $f_P(\lambda) := f(\lambda) + P$.

By (5.3) and Theorem 5.4(a), for each $j \in \{1, \ldots, n\}$ there exist $\delta_j \in (0, \delta_0]$ and $\alpha_j \in \mathbf{N} \cup \{0\}$ such that

$$\operatorname{nul}(f_{P_j}(\lambda)) = \alpha_j \leq \operatorname{nul}(f_{P_j}(\lambda_0)) \qquad (0 < |\lambda - \lambda_0| < \delta_j).$$

By (5.4), $\operatorname{nul}(f_P(\lambda)) = \operatorname{nul}(f_P(\lambda_0)) = 0$ for all $P \in \Pi(A)\backslash\{P_1, \ldots, P_n\}$ and all $\lambda \in K_{\delta_0}(\lambda_0)$. Put $\delta = \min\{\delta_1, \ldots, \delta_n\}$. Then we have

$$\nu(f(\lambda))(P_j) = \alpha_j \leq \nu(f(\lambda_0))(P_j) \qquad (j = 1, \ldots, n)$$

and

$$\nu(f(\lambda))(P) = \nu(f(\lambda_0))(P) = 0 \qquad (P \in \Pi(A)\backslash\{P_1, \ldots, P_n\})$$

for all $\lambda \in \dot{K}_\delta(\lambda_0)$.

(b) The proof is similar. ■

Now we are in a position to present the main results of this paper.

5.6 Theorem. *Let K be an inessential ideal of A, and let $f : G \to A$ be holomorphic.*

(a) *Suppose $f(\lambda) \in \Phi_l(A, K)$ for all $\lambda \in G$. Then there exists a discrete subset M_α of G such that*

(i) *$\nu(f(\lambda))$ is independent of λ for $\lambda \in G\backslash M_\alpha$,*

(ii) *for each $\mu \in M_\alpha$ there is a primitive ideal P such that*

$$\nu(f(\mu))(P) > \nu(f(\lambda))(P) \qquad (\lambda \in G\backslash M_\alpha).$$

(b) *Suppose $f(\lambda) \in \Phi_r(A, K)$ for all $\lambda \in G$. Then there exists a discrete subset M_β of G such that*

(i) $\delta(f(\lambda))$ *is independent of* λ *for* $\lambda \in G \backslash M_\beta$,
(ii) *for each* $\mu \in M_\beta$ *there is a primitive ideal* P *such that*

$$\delta(f(\mu))(P) > \delta(f(\lambda))(P) \quad (\lambda \in G \backslash M_\beta).$$

PROOF. (a) Let M_α be the set of points $\mu_0 \in G$ with the following property: there exists some neighbourhood $U \subseteq G$ of μ_0 such that with some constant $\gamma \geqslant 0$ and with some primitive ideal P the following assertion holds:

$$\nu(f(\lambda))(P) = \gamma < \nu(f(\mu_0))(P) \text{ for } \lambda \in U \backslash \{\mu_0\}.$$

Take $\mu_0 \in M_\alpha$. By Theorem 5.5(a), there exists $\delta > 0$ such that $\nu(f(\lambda))$ is independent of λ for $0 < |\lambda - \mu_0| < \delta$. Thus M_α is a discrete subset of G. Put $G_0 = G \backslash M_\alpha$. Observe that G_0 is a region.

Let $\mu \in G_0$. By Theorem 5.5(a), there exists $\delta > 0$ with

$$P \in \Pi(A) \Rightarrow \nu(f(\lambda))(P) \text{ is constant in } K_\delta(\mu). \tag{5.5}$$

Fix $\lambda_0 \in G_0$ and define

$$G_1 = \{\mu \in G_0 : \nu(f(\mu))(P) = \nu(f(\lambda_0))(P) \text{ for all } P \in \Pi(A)\},$$
$$G_2 = G_0 \backslash G_1.$$

From (5.5) we obtain that G_1 and G_2 are open subsets of G_0. Since G_0 is connected and $\lambda_0 \in G_1$, it follows that $G_2 = \emptyset$. Hence

$$G_1 = G_0 = G \backslash M_\alpha.$$

This proves (i). The definition of M_α shows that (ii) holds.
(b) The proof is similar. ■

5.7 Corollary. *Let K be an inessential ideal. Suppose that $f: G \to A$ is a holomorphic function with $f(\lambda) \in \Phi(A, K)$ for all $\lambda \in G$. Then $\iota(f(\lambda)) = \iota(f(\mu))$ for λ, $\mu \in G$. Furthermore there exists a discrete subset M of G such that*
(i) $\nu(f(\lambda)) = \nu(f(\mu))$ *and* $\delta(f(\lambda)) = \delta(f(\mu))$ *for* λ, $\mu \in G \backslash M$;
(ii) *for each* $\mu \in M$ *there is a primitive ideal* P *such that*

$$\nu(f(\mu))(P) > \nu(f(\lambda))(P) \text{ and } \delta(f(\mu))(P) > \delta(f(\lambda))(P) \qquad (\lambda \in G \backslash M).$$

PROOF. Define the sets M_α and M_β as in Theorem 5.6. By Proposition 5.1(a), $\iota(f(\lambda))(P)$ $(P \in \Pi(A))$ is constant in G. This shows $M_\alpha = M_\beta$. Put $M = M_\alpha$ $(=M_\beta)$. Then (i) is valid. To prove (ii), use again the continuity of the index. ■

5.8 Corollary. *Let K be an inessential ideal, and let $f: G \to A$ be holomorphic. Suppose that $f(\lambda) \in \Phi_l(A, K)[\Phi_r(A, K), \Phi(A, K)]$ for all $\lambda \in G$ and that $f(\lambda_0)$ is left invertible [right invertible, invertible] for some $\lambda_0 \in G$. Then there exist a discrete subset M of G and a holomorphic function $g: G \backslash M \to A$ such that*

$$g(\lambda)f(\lambda) = e\,[f(\lambda)g(\lambda) = e, f(\lambda)g(\lambda) = g(\lambda)f(\lambda) = e]\ \textit{for all } \lambda \in G\backslash M.$$

PROOF. We assume that $f(\lambda_0)$ is left invertible. By Proposition 4.6, we have $\nu(f(\lambda_0))(P) = 0$ for all $P \in \Pi(A)$. Theorem 5.6(a) shows

$$\nu(f(\lambda))(P) = 0 \text{ for all } \lambda \in G\backslash M_\alpha \text{ and all } P \in \Pi(A).$$

Put $M = M_\alpha$. It follows that $f(\lambda)$ is left invertible for all $\lambda \in G\backslash M$ (Proposition 4.6). The existence of a holomorphic $g : G\backslash M \to A$ with $g(\lambda)f(\lambda) = e$ follows from [1, theorem 1]. ■

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