A NOTE ON THE DECOMPOSITION OF FREDHOLM OPERATORS

CHRISTOPH SCHMOEGER

Abstract. We generalize a result of F. Szigeti concerning the decomposition of Fredholm operators on Banach spaces.

1. Terminology and results

Throughout this paper let X denote an *infinite-dimensional* Banach space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $\mathcal{L}(X)$ denote the Banach algebra of all bounded linear operators on X, and $\mathcal{L}(X)^{-1}$ the group of invertible operators in $\mathcal{L}(X)$. We write I for the identity on X.

For $A \in \mathcal{L}(X)$ set

$$\alpha(A) = \dim N(A) \quad \text{and} \quad \beta(A) = \operatorname{codim} A(X)\,,$$

where $N(A)$ is the kernel and $A(X)$ is the range of A. The operator A is called a *Fredholm operator* if both $\alpha(A)$ and $\beta(A)$ are finite. In this case the *index* $\operatorname{ind}(A)$ of A is defined by

$$\operatorname{ind}(A) = \alpha(A) - \beta(A)\,.$$

We denote by $\Phi(X)$ the set of all Fredholm operators on X.

Observe that $\mathcal{L}(X)^{-1} \subseteq \Phi(X)$. It follows from [2, Satz 55.4] that if $A \in \Phi(X)$, then $A(X)$ is closed.

For $k \in \mathbb{Z}$ let

$$\Phi_k(X) = \{A \in \Phi(X) : \operatorname{ind}(A) = k\}\,.$$

It is well-known that $\Phi(X)$ is open and if $\mathcal{C}$ is a component of $\Phi(X)$ then $\mathcal{C}$ is contained in $\Phi_k(X)$ for some $k \in \mathbb{Z}$.

1.1. Proposition. *Let $A, B \in \mathcal{L}(X)$.*
(1) *If $AB \in \Phi(X)$ and $A \in \Phi(X)$ [resp. $B \in \Phi(X)$], then $B \in \Phi(X)$ [resp. $A \in \Phi(X)$].*
(2) *If $A, B \in \Phi(X)$, then $AB \in \Phi(X)$ and*

$$\operatorname{ind}(AB) = \operatorname{ind}(A) + \operatorname{ind}(B)\,.$$

Proof. [2, Satz 71.2 and Satz 71.3] □

Part (2) of Proposition 1.1 is called the *index theorem*, which can be expressed as

$$\Phi_k(X) \cdot \Phi_j(X) \subseteq \Phi_{k+j}(X) \qquad (k, j \in \mathbb{Z})\,.$$

F. Szigeti has shown in [5] the following result:

Date: 13th June 2007.
1991 *Mathematics Subject Classification.* 47A53.
Key words and phrases. Fredholm operator, index.

1.2. Theorem. *Let X be a separable Hilbert space (with $\dim X = \infty$) and let $k, j \in \mathbb{Z}$. Then*

$$\Phi_k(X) \cdot \Phi_j(X) = \Phi_{k+j}(X) \Longleftrightarrow k \geq 0 \text{ or } j \leq 0 .$$

The aim of this note is a generalization of Theorem 1.2 to a large class of Banach spaces, which include separable Hilbert spaces.

We say that the Banach space has property $(*)$, if

$$(*) \qquad X \text{ is isomorphic to } X \oplus \mathbb{K} .$$

The familiar Banach spaces, as $l^p, L^p, (c), (c_0), C(K), \dots$ have property $(*)$. But there are Banach spaces which have not property $(*)$. For examples see [1], [3], and [4].

Observe that if X has property $(*)$, then

$$X \text{ is isomorphic to } X \oplus \mathbb{K}^n \quad \text{for all } n \in \mathbb{N} .$$

Our generalization of Theorem 1.2 reads as follows:

1.3. Theorem. *Suppose that X has property $(*)$ and suppose that $k, j \in \mathbb{Z}$. Then*

$$\Phi_k(X) \cdot \Phi_j(X) = \Phi_{k+j}(X) \Longleftrightarrow k \geq 0 \text{ or } j \leq 0 .$$

A proof of this result will be given in Section 2 of this paper.

An immediate consequence of Theorem 1.3 is:

1.4. Corollary. *Let X have property $(*)$ and let $k, j \in \mathbb{Z}$.*

(1) *If $k \geq 0$ and $j \geq 0$, then*

$$\Phi_k(X) \cdot \Phi_j(X) = \Phi_j(X) \cdot \Phi_k(X) = \Phi_{k+j}(X) .$$

(2) *If $k \leq 0$ and $j \leq 0$, then*

$$\Phi_k(X) \cdot \Phi_j(X) = \Phi_j(X) \cdot \Phi_k(X) = \Phi_{k+j}(X) .$$

(3) *If $k \geq 0$ and $j \leq 0$, then*

$$\Phi_k(X) \cdot \Phi_j(X) = \Phi_{k+j}(X) .$$

(4) *If $k \leq 0$ and $j \geq 0$, then*

$$\Phi_j(X) \cdot \Phi_k(X) = \Phi_{k+j}(X) .$$

(5) $\Phi_k(X) = \Phi_{|k|}(X) \cdot \Phi_{k-|k|}(X)$.

A proof of the following characterization of Banach spaces with property $(*)$ will be given in Section 2 of this paper.

1.5. Theorem. *The following assertions are equivalent:*

(1) *X has property $(*)$;*
(2) *$\Phi_k(X) \cdot \Phi_j(X) = \Phi_{k+j}(X)$ for all $k, j \in \mathbb{Z}$ such that $k \geq 0$ or $j \leq 0$;*
(3) *$\Phi_0(X) = \Phi_1(X) \cdot \Phi_{-1}(X)$;*
(4) *$\mathcal{L}(X)^{-1} \subseteq \Phi_1(X) \cdot \Phi_{-1}(X)$;*
(5) *$I \in \Phi_1(X) \cdot \Phi_{-1}(X)$.*

2. Proofs and corollaries

For the proofs of Theorem 1.3 and Theorem 1.5 we need some preparations.

Notation. Let $\mathbb{N} = \{1, 2, 3 \cdots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2.1. Proposition. *Let X have property $(*)$ and let $n \in \mathbb{N}$. Then there is $B \in \Phi(X)$ such that*

$$\alpha(B) = n \quad \textit{and} \quad \beta(B) = 0\,.$$

Proof. Since X is isomorphic to $X \oplus \mathbb{K}^n$, there is a bounded linear bijection

$$\Psi_1 : X \oplus \mathbb{K}^n \to X\,.$$

Define the bounded linear mapping $\Psi_2 : X \oplus \mathbb{K}^n \to X$ by

$$\Psi_2(x; \alpha_1, \ldots, \alpha_n) = x \qquad (x \in X,\ \alpha_1, \ldots, \alpha_n \in \mathbb{K})$$

and the operator $B \in \mathcal{L}(X)$ by

$$B = \Psi_2 \Psi_1^{-1}\,.$$

Then

$$B(X) = \Psi_2(\Psi_1^{-1}(X)) = \Psi_2(X \oplus \mathbb{K}^n) = X\,,$$

hence $\beta(B) = 0$. Furthermore we have

$$N(B) = \{\Psi_1(0; \alpha_1, \ldots, \alpha_n) : \alpha_1, \ldots, \alpha_n \in \mathbb{K}\}\,,$$

hence $\alpha(B) = n$, since Ψ_1 is bijective. □

2.2. Proposition. *The following assertions are equivalent:*
(1) *X has property $(*)$;*
(2) *there is $B \in \Phi(X)$ with*

$$\alpha(B) = 1 \quad \textit{and} \quad \beta(B) = 0\,.$$

Proof. Proposition 2.1 shows that (1) implies (2). Now suppose that (2) holds. There is $x_0 \in X \setminus \{0\}$ such that $N(B) = \{\alpha x_0 : \alpha \in \mathbb{K}\}$. Hence there is a closed subspace X_1 of X with

$$X = N(B) \oplus X_1\,.$$

Then $X = B(X) = B(X_1)$. Let $B_0 = B_{|X_1}$, hence $B_0 : X_1 \to X$ is bijective.

Now define the bounded linear mapping $\Psi : X \oplus \mathbb{K} \to X$ by

$$\Psi(x; \alpha) := B_0^{-1} x + \alpha x_0 \qquad (x \in X,\ \alpha \in \mathbb{K})\,.$$

It is easy to see that Ψ is bijective, hence X has property $(*)$. □

Proof of Theorem 1.3. "$\Rightarrow$": We assume that $\Phi_k(X) \cdot \Phi_j(X) = \Phi_{k+j}(X)$ and have to show that $k \geq 0$ or $j \leq 0$. Assume to the contrary that $k < 0$ and $j > 0$.

Case 1: $k + j = 0$. Then $I \in \Phi_{k+j}(X)$, hence $I = A_1 A_2$ with $A_1 \in \Phi_k(X)$ and $A_2 \in \Phi_j(X)$. Since $N(A_2) = \{0\}$, it follows that

$$0 = \alpha(A_2) = \beta(A_2) + j \geq j > 0\,,$$

a contradiction.

Case 2: $k + j > 0$. By Proposition 2.1, there is $A \in \Phi(X)$ such that $\alpha(A) = k + j$ and $\beta(A) = 0$, hence $A \in \Phi_{k+j}(X)$. Then $A = A_1 A_2$ with $A_1 \in \Phi_k(X)$ and $A_2 \in \Phi_j(X)$. Since $N(A_2) \subseteq N(A_1 A_2), \alpha(A_1 A_2) \geq \alpha(A_2)$, thus

$$\alpha(A_2) = \beta(A_2) + j \geq j > j + k = \alpha(A) = \alpha(A_1 A_2) \geq \alpha(A_2)\,,$$

a contradiction.

Case 3: $k+j<0$. Proposition 2.1 shows that there is $A \in \Phi(X)$ with $\alpha(A) = -(k+j)$ and $\beta(A) = 0$. Hence $A \in \Phi_{-(k+j)}(X)$. Because of $\beta(A) = 0$, there is $B \in \mathcal{L}(X)$ with $AB = I$ (see [2, Satz 74.3]). From Proposition 1.1 we get $B \in \Phi(X)$ and

$$0 = \text{ind}\,(AB) = \text{ind}\,(A) + \text{ind}\,(B)\,,$$

hence $B \in \Phi_{k+j}(X)$ and $\alpha(B) = 0$. Therefore B has the decomposition $B = B_1 B_2$ with $B_1 \in \Phi_k(X)$ and $B_2 \in \Phi_j(X)$. We derive

$$\beta(B_1) \quad = \alpha(B_1) - k \geq -k > -k - j = \text{ind}\,(A) = -\text{ind}\,B = \beta(B)\,.$$

Since $(B_1B_2)(X) \subseteq B_1(X)$, we have $\beta(B_1B_2) \geq \beta(B_1)$, thus

$$\beta(B_1) > \beta(B) = \beta(B_1B_2) \geq \beta(B_1)\,,$$

a contradiction.

"$\Leftarrow$": Now assume that $k \geq 0$ or $j \leq 0$.

Case 1: $k \geq 0$. By Proposition 2.1, there is $B \in \Phi(X)$ with $\alpha(B) = k$ and $\beta(B) = 0$. From [2, Satz 74.3] we see that there is $C \in \mathcal{L}(X)$ such that $BC = I$. Proposition 1.1 gives $C \in \Phi(X)$ and $\text{ind}\,(C) = -\text{ind}\,(B)$. Now take $A \in \Phi_{k+j}(X)$ and let $A_1 = B$ and $A_2 = CA$. Thus

$$A_1, A_2 \in \Phi(X) \quad \text{and} \quad A = A_1A_2\,.$$

We have

$$\text{ind}\,(A_1) = \text{ind}\,(B) = \alpha(B) = k$$

and

$$\text{ind}\,(A_2) = \text{ind}\,(CA) = \text{ind}\,(C) + \text{ind}\,(A) = -\text{ind}\,(B) + \text{ind}(A) = -k + k + j = j\,,$$

thus $A \in \Phi_k(X) \cdot \Phi_j(X)$.

Case 2: $j \leq 0$. By Proposition 2.1, there is $B \in \Phi(X)$ with $\alpha(B) = -j$ and $\beta(B) = 0$. As in *Case 1*, $BC = I$ for some $C \in \mathcal{L}(X)$. Proposition 1.1 gives $C \in \Phi(X)$ and $\text{ind}\,(C) = -\text{ind}\,(B)$. Now let $A \in \Phi_{k+j}(X)$ and let $A_1 = AB$ and $A_2 = C$. Then

$$A_1, A_2 \in \Phi(X) \quad \text{and} \quad A = A_1A_2\,.$$

We have

$$\text{ind}\,(A_1) = \text{ind}\,(AB) = \text{ind}\,(A) + \text{ind}\,(B) = k + j - j = k\,,$$

and

$$\text{ind}\,(A_2) = \text{ind}\,(C) = -\text{ind}\,(B) = -\alpha(B) = j\,,$$

therefore $A \in \Phi_k(X) \cdot \Phi_j(X)$. □

Proof of Theorem 1.5. Theorem 1.3 shows that (1) implies (2). The implications (2) ⇒ (3) ⇒ (4) ⇒ (5) are clear. Now let (5) hold. Then there are $A_1 \in \Phi_1(X)$ and $A_2 \in \Phi_2(X)$ such that $I = A_1A_2$. It follows that $\beta(A_1) = 0$ and

$$\alpha(A_1) = \alpha(A_1) - \beta(A_1) = \text{ind}\,(A_1) = 1\,,$$

hence (1) holds, by Proposition 2.2. □

The proof of Theorem 1.3 shows the following:

2.3. Corollary. *Let X have property $(*)$ and let $k, j \in \mathbb{Z}$.*

(1) *If $k \geq 0$, then there are $B, C \in \Phi(X)$ such that*

$$BC = I, \ \mathrm{ind}\,(B) = \alpha(B) = k, \ \mathrm{ind}\,(C) = \beta(C) = -k,$$
$$A = B(CA) \quad \textit{for each } A \in \Phi_{k+j}(X)\,,$$

and

$$\mathrm{ind}\,(CA) = j \quad \textit{for each } A \in \Phi_{k+j}(X)\,,$$

hence

$$\Phi_{k+j}(X) = B \cdot \Phi_j(X)\,.$$

(2) *If $j \leq 0$, then there are $B, C \in \Phi(X)$ such that*

$$BC = I, \ \mathrm{ind}\,(B) = \alpha(B) = -j, \ \mathrm{ind}\,(C) = \beta(C) = j,$$
$$A = (AB)C \quad \textit{for each } A \in \Phi_{k+j}(X)\,,$$

and

$$\mathrm{ind}\,(AB) = k \quad \textit{for each } A \in \Phi_{k+j}(X)\,,$$

hence

$$\Phi_{k+j}(X) = \Phi_k(X) \cdot C\,.$$

2.4. Corollary. *Let X have property $(*)$ and let $n \in \mathbb{N}_0$.*

(1) *There is $B \in \Phi(X)$ with*

$$\mathrm{ind}\,(B) = \alpha(B) = 1$$

and

$$\Phi_{n+1}(X) = B^n \cdot \Phi_1(X)\,.$$

(2) *There is $C \in \Phi(X)$ with*

$$\mathrm{ind}\,(C) = \beta(C) = -1$$

and

$$\Phi_{-(n+1)}(X) = \Phi_{-1}(X) \cdot C^n\,.$$

(3) *If B and C are as in (1) and (2), then*

$$\Phi_0(X) = B \cdot \Phi_{-1}(X) = \Phi_1(X) \cdot C\,.$$

Proof. (1) With $k = 1$ let B as in Corollary 2.3 (1), hence, with $j = 1, \Phi_2(X) = B \cdot \Phi_1(X)$. Then

$$\Phi_3(X) \quad = \Phi_2(X) \cdot \Phi_1(X) = B \cdot \Phi_1(X) \cdot \Phi_1(X) = B \cdot \Phi_2(X) = B^2 \Phi_1(X)\,.$$

Now proceed with induction.

(2) Similar, use Corollary 2.3 (2).

(3) Use Corollary 2.3 (1) and (2) with $k = 1$ and $j = -1$. □

Case 1 in the first part of the proof of Theorem 1.3 makes no use of property $(*)$, hence we have:

2.5. Corollary. *If X is an arbitrary Banach space, then*

$$I \notin \Phi_{-n}(X) \cdot \Phi_n(X) \quad \textit{for all } n \in \mathbb{N},$$

thus

$$\Phi_{-n}(X) \cdot \Phi_n(X) \underset{\neq}{\subset} \Phi_0(X) \quad (n \in \mathbb{N}).$$

2.6. Remark. Let X, k and j as in Corollary 2.3. The decomposition of an operator $A \in \Phi_{k+j}(X)$ is not unique. For example let $k \geq 0$ and let B and C as in Corollary 2.3 (1). Then $\alpha(B) = k$ and $\beta(B) = 0$. Satz 82.4 in [2] shows that there is $\delta > 0$ such that if $S \in \mathcal{L}(X)$ and $\|S\| < \delta$, then

$$B + S \in \Phi(X), \text{ind}\,(B+S) = \text{ind}\,(B)$$
$$\alpha(B+S) \leq \alpha(B) \quad \text{and} \quad \beta(B+S) \leq \beta(B).$$

Take $S \in \mathcal{L}(X)$ with $\|S\| < \delta$. Then it follows that

$$\alpha(B+S) = k \quad \text{and} \quad \beta(B+S) = 0.$$

As in the proof of Theorem 1.3, $B+S$ is right invertible, thus $(B+S)C_0 = I$ for some $C_0 \in \Phi(X)$ with $\text{ind}\,(C_0) = -k$. For each $A \in \Phi_{k+j}(X)$ we therefore have the decompositions

$$A = B(CA) \quad \text{and} \quad A = (B+S)(C_0 A)$$

with

$$B, B+S \in \Phi_k(X) \quad \text{and} \quad CA, C_0 A \in \Phi_j(X).$$

The author thanks Dr. G. Herzog for some helpful discussions.

References

[1] W. T. Gowers: *A solution to Banach's hyperplane problem.* Bull. London Math. Soc. 26 (1994), 523–530.
[2] H. Heuser: *Funktionalanalysis, 3. ed.* Teubner (1991).
[3] N. Kalton, T. Peck: *Twisted sums of sequence spaces and the three space problem.* Trans. Amer. Math. Soc. 255 (1979), 1–30.
[4] S. Rolewicz: *An example of a normed space non-isomorphic to its product by the real line.* Studia Math. 40 (1971), 71–75.
[5] F. Szigeti: *Decomposition of Fredholm operators.* Publ. Math. Debrecen 19 (1972), 187–189.

Christoph Schmoeger
Institut für Analysis
Universität Karlsruhe (TH)
Englerstrasse 2
76128 Karlsruhe
Germany
E-mail address: christoph.schmoeger@math.uni-karlsruhe.de
URL: http://www.mathematik.uni-karlsruhe.de/mi1weis/ schmoeger/