SPECTRAL RADII OF GENERALIZED INVERSES OF SIMPLY POLAR MATRICES

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Abstract. In this note we study the spectral radii of generalized inverses of square matrices A such that $\operatorname{rank}(A) = \operatorname{rank}(A^2)$.

1. General and introductory material

For positive integers n and m, $\mathbb{C}^{n\times m}$ denotes the vector space of all complex $n \times m$ matrices.
Let $A \in \mathbb{C}^{n\times n}$ be a square matrix. A matrix $C \in \mathbb{C}^{n\times n}$ is called a g_1-*inverse* of A if

$$ACA = A\,.$$

If $B \in \mathbb{C}^{n\times n}$ and

$$ABA = A \quad \text{and} \quad BAB = B\,,$$

then B is called a g_2-*inverse* of A. By $\mathscr{G}_1(A)$ we denote the set of all g_1-inverses of A. $\mathscr{G}_2(A)$ is the set of all g_2-inverses of A. It is well-known that $\mathscr{G}_1(A) \neq \phi$ (see [1]). Furthermore it is easy to see that if $C \in \mathscr{G}_1(A)$, then $B = CAC \in \mathscr{G}_2(A)$, hence

$$\phi \neq \mathscr{G}_2(A) \subseteq \mathscr{G}_1(A)\,.$$

If A is non-singular, then $\mathscr{G}_2(A) = \mathscr{G}_1(A) = \{A^{-1}\}$.

For $A \in \mathbb{C}^{n\times n}$ we denote the set of eigenvalues of A by $\sigma(A)$ and the *spectral radius* $r(A)$ of A is defined by

$$r(A) = \max_{\lambda\in\sigma(A)} |\lambda|\,.$$

Let $A \in \mathbb{C}^{n\times m}$. A^T denotes the transpose of A and A^* denotes the conjugate transpose of A. The *range* of A is given by

$$\mathscr{R}(A) = \{Ax : x \in \mathbb{C}^n\}$$

and the *kernel* of A is the set

$$\mathscr{N}(A) = \{x \in \mathbb{C}^n : Ax = 0\}$$

(we follow the convention $\mathbb{C}^n = \mathbb{C}^{n\times 1}$).

In this note we study the set

$$R_A = \{r(C) : C \in \mathscr{G}_1(A)\}$$

for $A \in \mathbb{C}^{n\times n}$ such that $\operatorname{rank}(A) = \operatorname{rank}(A^2)$, where $\operatorname{rank}(A) = \dim \mathscr{R}(A)$. Such matrices are called *simply polar*.

Examples. If A is non-singular, then $R_A = \{r(A)^{-1}\}$. If $A = 0$, then $ACA = A$ for each $C \in \mathbb{C}^{n\times n}$, hence $R_A = [0, \infty)$.

Throughout this paper we will assume that $n \geq 2$. The identity on $\mathbb{C}^n$ is denoted by I_n.

Date: 20th August 2007.
1991 *Mathematics Subject Classification.* 15 A 09.
Key words and phrases. simply polar matrix, generalized inverse.

1.1. Proposition. *If $A \in \mathbb{C}^{n\times n}$ and $B \in \mathscr{G}_2(A)$, then*

$$\mathscr{G}_1(A) = \{B + T - BATAB : T \in \mathbb{C}^{n\times n}\}$$

Proof. [1, Theorem 2 in Chapter 2.3]. □

It follows from Proposition 1.1, that if A is singular, then $\mathscr{G}_1(A)$ is an infinite set. In [6], the following result is shown:

1.2. Proposition. *Suppose that $A \in \mathbb{C}^{n\times n}$ is singular. We have:*
(1) *for each $z \in \mathbb{C}$, there is $B \in \mathscr{G}_1(A)$ with $z \in \sigma(B)$;*
(2) *if $B \in \mathscr{G}_2(A)$, then*

$$B + z(I_n - BA),\ B + z(I_n - AB) \in \mathscr{G}_1(A)$$

for all $z \in \mathbb{C}$ and

$$r(B + z(I_n - BA)) = r(B + z(I_n - AB)) = \begin{cases} r(B), & \text{if } |z| \le r(B) \\ |z|, & \text{if } |z| > r(B); \end{cases}$$

(3) *$[r(B), \infty) \subseteq R_A$ for each $B \in \mathscr{G}_2(A)$.*

1.3. Proposition. *Suppose that $A \in \mathbb{C}^{n\times n}$, $r = \operatorname{rank}(A) > 0$ and that A has a decomposition*

$$A = U \left[\begin{array}{c:c} D & 0 \\ \hdashline 0 & 0 \end{array}\right] V^{-1}$$

with $U, V \in \mathbb{C}^{n\times n}$ non-singular and $D \in \mathbb{C}^{r\times r}$ non-singular. Then

$$B = V \left[\begin{array}{c:c} D^{-1} & 0 \\ \hdashline 0 & 0 \end{array}\right] U^{-1} \in \mathscr{G}_2(A)$$

and

$$\mathscr{G}_1(A) = \left\{ V \left[\begin{array}{c:c} D^{-1} & A_1 \\ \hdashline A_2 & A_3 \end{array}\right] U^{-1} : A_1 \in \mathbb{C}^{r\times(n-r)},\ A_2 \in \mathbb{C}^{(n-r)\times r},\ A_3 \in \mathbb{C}^{(n-r)\times(n-r)} \right\}.$$

Proof. It is easy to verify that $B \in \mathscr{G}_2(A)$. Let $T \in \mathbb{C}^{n\times n}$, let $\varphi(T) = V^{-1}TU$ and set $B_0 := B + T - BATAB$. Then

$$\begin{aligned} B_0 &= V \left[\begin{array}{c:c} D^{-1} & 0 \\ \hdashline 0 & 0 \end{array}\right] U^{-1} + T - V^{-1} \left[\begin{array}{c:c} I_r & 0 \\ \hdashline 0 & 0 \end{array}\right] \varphi(T) \left[\begin{array}{c:c} I_r & 0 \\ \hdashline 0 & 0 \end{array}\right] U^{-1} \\ &= V \left(\left[\begin{array}{c:c} D^{-1} & 0 \\ \hdashline 0 & 0 \end{array}\right] + \varphi(T) - \left[\begin{array}{c:c} I_r & 0 \\ \hdashline 0 & 0 \end{array}\right] \varphi(T) \left[\begin{array}{c:c} I_r & 0 \\ \hdashline 0 & 0 \end{array}\right] \right) U^{-1} \\ &= V \left[\begin{array}{c:c} D^{-1} & A_1 \\ \hdashline A_2 & A_3 \end{array}\right] U^{-1}. \end{aligned}$$

Since the mapping $\varphi : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ is bijective, the result follow from Proposition 1.1. □

Recall that a matrix $A \in \mathbb{C}^{n\times n}$ is called *simply polar* if $\operatorname{rank}(A) = \operatorname{rank}(A^2)$.

1.4. Proposition. *Let $A \in \mathbb{C}^{n\times n}$ be singular. The following assertions are equivalent:*
(1) *A is simply polar;*
(2) *0 is a simple pole of the resolvent $(\lambda I_n - A)^{-1}$;*
(3) *$\mathbb{C}^n = \mathscr{R}(A) \oplus \mathscr{N}(A)$;*
(4) *there is $B \in \mathscr{G}_2(A)$ such that $AB = BA$.*

Proof. [3, Satz 72.4], [3, Satz 101.2] and [1, Theorem 5.2]. □

If $A \in \mathbb{C}^{n\times n}$ is simply polar, then, by Proposition 1.4, there is $B \in \mathbb{C}^{n\times n}$ such that $ABA = A, BAB = B$ and $AB = BA$. It is shown in [1, Theorem 5.1], that there is no other g_2-inverse of A which commutes with A. B is called the *Drazin-inverse* of A. The following result is shown in [1, p. 53].

1.5. Proposition. *If $A \in \mathbb{C}^{n\times n}$, $A \neq 0$ and if A is simply polar, then the Drazin-inverse B of A satisfies*

$$\sigma(B) \setminus \{0\} = \left\{ \frac{1}{\lambda} : \lambda \in \sigma(A) \setminus \{0\} \right\}$$

and hence $r(B) = r(A)^{-1}$.

2. Generalized inverses of simply polar matrices

Throughout this section we assume that $A \in \mathbb{C}^{n\times n}$ is simply polar and that $\operatorname{rank}(A) > 0$.

By [5, 4.3.2 (4)] (see also [4]), A has a decomposition

$$A = U \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array} \right] U^{-1}, \tag{2.1}$$

wher $U \in \mathbb{C}^{n\times n}$ and $D \in \mathbb{C}^{r\times r}$ are non singular. From Proposition 1.3 we know that

$$B = U \left[\begin{array}{c|c} D^{-1} & 0 \\ \hline 0 & 0 \end{array} \right] U^{-1} \in \mathscr{G}_2(A). \tag{2.2}$$

It is easy to see that the matrix B in (2.2) is the Drazin-invers of A.

2.1. Theorem. *The following assertions are equivalent:*
(1) $\dim \mathscr{N}(A) \geq \operatorname{rank}(A)$.
(2) there is $B \in \mathscr{G}_2(A)$ with $B^2 = 0$.

A consequence of Theorem 2.1 is:

2.2. Corollary. *If* $\dim \mathscr{N}(A) \geq \operatorname{rank}(A)$, *then there is an entire function $F : \mathbb{C} \to \mathbb{C}^{n\times n}$ such that*

$$F(z) \in \mathscr{G}_1(A),\ \sigma(F(z)) = \{z, 0\} \text{ and } \ r(F(z)) = |z| \quad \text{for all } z \in \mathbb{C}.$$

Furthermore we have $R_A = [0, \infty)$.

Proof. By Theorem 2.1, there is $B \in \mathscr{G}_2(A)$ with $B^2 = 0$. Define F by $F(z) = B + z(I_n - AB)$. Then $F(z) \in \mathscr{G}_1(A)$ for each $z \in \mathbb{C}$ (Proposition 1.2). [6, Theorem 3] gives

$$\{z\} \subseteq \sigma(F(z)) \subseteq \{z, 0\} \qquad (z \in \mathbb{C}).$$

Assume that $F(z)$ is non-singular for some $z \in \mathbb{C}$. Thus there is $C \in \mathbb{C}^{n\times n}$ with $F(z)C = I_n$. Since $BF(z) = 0$, we get $0 = BF(z)C = B$, thus $A = ABA = 0$, a contradiction. □

Proof of Theorem 2.1. Let $r = \operatorname{rank}(A)$.
(1) $\Rightarrow$ (2): Proposition 1.4 (3) shows that $n - r = \dim \mathscr{N}(A) \geq r$.
Case 1: $n - r = r$. Let D be as in (2.1) and let

$$S = \begin{bmatrix} D^{-1} & D^{-1} \\ -D^{-1} & -D^{-1} \end{bmatrix} \text{ and } B = USU^{-1}.$$

Then it is easy to see $B \in \mathscr{G}_2(A)$ and $B^2 = 0$.
Case 2: $n - r > r$. Then $r < n/2$.
Case 2.1: $n = 2m$ for some $m \in \mathbb{N}$. Let

$$T = \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{m \times m},\ S = \begin{bmatrix} T & T \\ -T & -T \end{bmatrix} \in \mathbb{C}^{n \times n}$$

and $B = USU^{-1}$. Then $B \in \mathscr{G}_2(A)$ and $B^2 = 0$.
Case 2.2: $n = 2m + 1$ for some $m \in \mathbb{N}$. Then $r < m$. Set

$$T = \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{m \times m},\ S = \begin{bmatrix} T & T & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ -T & -T & \\ 0 \cdots & \cdots 0 & 0 \end{bmatrix} \in \mathbb{C}^{n \times n}$$

and $B = USU^{-1}$. As above, $B \in \mathscr{G}_2(A)$ and $B^2 = 0$.
(2) $\Rightarrow$ (1): Since $B^2 = 0$, A is singular. We have $(BA)^2 = BA$, $\mathscr{R}(BA) = \mathscr{R}(B)$, $\mathscr{N}(A) = \mathscr{R}(I - BA)$, $\mathscr{R}(AB) = \mathscr{R}(A)$, $(AB)^2 = AB$ and

$$\mathbb{C}^n = \mathscr{R}(B) \oplus \mathscr{N}(A),$$

thus, by Proposition 1.4 (3), $\operatorname{rank}(B) = r = \operatorname{rank}(A)$. Now let $z \in \mathscr{R}(A) \cap \mathscr{R}(B)$. Then $z = ABz = BAz$, therefore $z = AB^2Az = 0$. This gives $\mathscr{R}(A) \cap \mathscr{R}(B) = \{0\}$. Since

$$\mathscr{R}(A) \oplus \mathscr{R}(B) \subseteq \mathbb{C}^n,$$

we derive $2r \leq n$, hence $\operatorname{rank}(A) = r \leq n - r = \dim \mathscr{N}(A)$. □

A square matrix D is said to be *non-derogatory* if its characteristic polynomial is also its minimal polynomial.

2.3. Theorem. *Suppose that* $\operatorname{rank}(A) = \operatorname{rank}(A^2) = n - 1$, *let* D *be as in (2.1) and suppose that* D *is non-derogatory. Then* A *has a nilpotent* g_1*-inverse and hence* $\min R_A = 0$.

Proof. Since D^{-1} is also non-derogatory, it follows from [2, Theorem 3.4] that there are $a_1 \in \mathbb{C}^{n-1}$, $a_2 \in \mathbb{C}^{n-1}$ and $a_3 \in \mathbb{C}$ such that

$$S = \begin{bmatrix} D^{-1} & a_1 \\ a_2^T & a_3 \end{bmatrix} \text{ is nilpotent},$$

hence $S^q = 0$ for some positive integer q. Let $B = USU^{-1}$. Then $B^q = 0$. By Proposition 1.3, $B \in \mathscr{G}_1(A)$. □

A matrix $N \in \mathbb{C}^{n \times n}$ is called *normal* if $NN^* = N^*N$. The spectral theorem for normal matrices implies that

$$N = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} U^*, \tag{2.3}$$

with $U \in \mathbb{C}^{n \times n}$ unitary (that is $UU^* = U^*U = I_n$) and $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_r)$, where $\lambda_1, \ldots, \lambda_r$ are the non-zero eigenvalues of N. It follows (see [5, 4.3.2 (4)]) that N is simply polar.
Now suppose that $\operatorname{rank}(N) = n - 1$. If $\lambda_i \neq \lambda_j$ $(i \neq j; i, j = 1, \ldots, n - 1)$ then the matrix D in (2.3) is

non-derogatory.

Thus we have proved:

2.4. Corollary. *If $N \in \mathbb{C}^{n\times n}$ is normal,* $\operatorname{rank}(N) = n-1$ *and if $\lambda_i \neq \lambda_j$ $(i \neq j; i,j = 1,\ldots,n-1)$ for the non-zero eigenvalues of N, then there is a nilpotent g_1-inverse of A.*

3. The case $n = 2$

3.1. Proposition. *If $A \in \mathbb{C}^{2\times 2}$ and $A^2 = 0$, then there is $B \in \mathscr{G}_2(A)$ such that $B^2 = 0$.*

Proof. The Schur decomposition of A is

$$A = U \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} U^*,$$

where $U \in \mathbb{C}^{2\times 2}$ is unitary and $\alpha \in \mathbb{C}$ (see [5, 5.2.3 (1)]. If $\alpha = 0$, we are done. So assume that $\alpha \neq 0$. Let

$$B = U \begin{bmatrix} 0 & 0 \\ \alpha^{-1} & 0 \end{bmatrix} U^*.$$

then it is easy to see that $B \in \mathscr{G}_2(A)$ and $B^2 = 0$. □

3.2. Theorem. *Suppose that $A \in \mathbb{C}^{2\times 2}$ is singular. Then there is $B \in \mathscr{G}_2(A)$ with $B^2 = 0$ and hence $R_A = [0,\infty)$.*

Proof. Because of Proposition 3.1, we assume that $A^2 \neq 0$. Since A is singular, we have $\operatorname{rank}(A) = \operatorname{rank}(A^2) = 1$, A is simply polar and $\dim \mathscr{N}(A) = \operatorname{rank}(A)$. Theorem 2.1 gives the result. □

4. Generalized inverses of projections

In this section we assume that $P \in \mathbb{C}^{n\times n}$, $0 \neq P \neq I_n$ and $P^2 = P$. Hence P is simply polar.

Since $\mathscr{R}(P) = \{x \in \mathbb{C}^n : Px = x\}$, it follows that $\sigma(P) = \{0,1\}$ and that there is a non-singular $U \in \mathbb{C}^{n\times n}$ such that

$$P = U \left[\begin{array}{c:c} I_r & 0 \\ \hdashline 0 & 0 \end{array}\right] U^{-1} \qquad (4.1)$$

([5, 9.8 (3)]), where $r = \operatorname{rank}(P)$.
From Theorem 2.1 we know that

$$\dim \mathscr{N}(P) \geq \operatorname{rank}(P) \Leftrightarrow \text{there is } B \in \mathscr{G}_2(P) \text{ such that } B^2 = 0\,.$$

So it remains to investigate the case where $\dim \mathscr{N}(P) < \operatorname{rank}(P)$:

4.1. Theorem. *If* $\dim \mathscr{N}(P) < \operatorname{rank}(P)$ *and if $B \in \mathscr{G}_1(P)$, then $1 \in \sigma(B)$ and hence $r(B) \geq 1$.*

Proof. Proposition 1.3 and (4.1) show that there are $A_1 \in \mathbb{C}^{r\times(n-1)}$, $A_2 \in \mathbb{C}^{(n-r)\times r}$ and $A_3 \in \mathbb{C}^{(n-r)\times(n-r)}$ such that

$$B = U \left[\begin{array}{c:c} I_r & A_1 \\ \hdashline A_2 & A_3 \end{array}\right] U^{-1}.$$

Denote by $a^{(1)}, \ldots, a^{(r)}$ the columns of A_2. Since

$$\operatorname{rank}(A_2) \leq n - r = \dim \mathscr{N}(A) < \operatorname{rank}(P) = r,$$

there is $(\alpha_1, \ldots, \alpha_r)^T \in \mathbb{C}^r$ such that $(\alpha_1, \ldots, \alpha_r) \neq 0$ and

$$\alpha_1 a^{(1)} + \cdots + \alpha_r a^{(r)} = 0 .$$

Set $x = (\alpha_1, \ldots, \alpha_r, 0, \ldots, 0)^T \in \mathbb{C}^n$ and $z = Ux$, then $z \neq 0$ and

$$Bz = U \left[\begin{array}{c:c} I_r & A_1 \\ \hdashline A_2 & A_3 \end{array} \right] x = Ux = z ,$$

thus $1 \in \sigma(B)$. □

4.2. Corollary.
(1) $R_P = [0, \infty) \Leftrightarrow \dim \mathscr{N}(P) \geq \operatorname{rank}(P)$.
(2) $R_P = [1, \infty) \Leftrightarrow \dim \mathscr{N}(P) < \operatorname{rank}(P)$.

Proof. (1) Theorem 4.1 and Corollary 2.2. (2) Theorem 4.1 and Proposition 1.2 (3). Observe that $P \in \mathscr{G}_1(P)$ and $r(P) = 1$. □

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