

CHARACTERIZATIONS OF SOME CLASSES OF RELATIVELY REGULAR OPERATORS

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ABSTRACT. A bounded linear operator A on a Banach space is called relatively regular, if there is a bounded linear operator B such that $ABA = A$. In this case B is called a g_1 -inverse of A . In this paper we characterize some classes of relatively regular operators A via the set $\{B_1 - B_2 : B_1 \text{ and } B_2 \text{ are } g_1\text{-inverses of } A\}$.

1. Terminology and introduction

Throughout this paper let X denote an infinite-dimensional complex Banach space and let $\mathcal{L}(X)$ denote the Banach algebra of all bounded linear operators on X .

If $A \in \mathcal{L}(X)$, we denote by $N(A)$ the kernel and by $A(X)$ the range of A . We write $\sigma(A)$ for the spectrum of A . The *ascend* $a(A)$ and the *descent* $d(A)$ of A are defined by

$$a(A) = \inf\{n \geq 0 : N(A^n) = N(A^{n+1})\}$$

and

$$d(A) = \inf\{n \geq 0 : A^n(X) = A^{n+1}(X)\},$$

respectively (where $\inf \emptyset = \infty$).

We call $A \in \mathcal{L}(X)$ *relatively regular* if $ABA = A$ for some $B \in \mathcal{L}(X)$. In this case AB is a projection onto $A(X)$ and $I - BA$ is a projection onto $N(A)$ and we say that B is a g_1 -inverse of A .

The set of all relatively regular operators on X is denoted by $\mathcal{R}(X)$.

If $A \in \mathcal{R}(X)$ let

$$\mathcal{G}_1(A) = \{B \in \mathcal{L}(X) : B \text{ is a } g_1\text{-inverse of } A\}.$$

If $B \in \mathcal{G}_1(A)$, set $B_0 = BAB$. Then it is easy to see that

$$AB_0A = A \quad \text{and} \quad B_0AB_0 = B_0.$$

B_0 is called a g_2 -inverse of A . Set

$$\mathcal{G}_2(A) = \{B \in \mathcal{L}(X) : B \text{ is a } g_2\text{-inverse of } A\}.$$

Then

$$\emptyset \neq \mathcal{G}_2(A) \subseteq \mathcal{G}_1(A).$$

For $A \in \mathcal{R}(X)$ let

$$\mathcal{D}(A) = \{B_1 - B_2 : B_1, B_2 \in \mathcal{G}_1(A)\}.$$

In this paper we characterize the following classes of relatively regular operators A via the sets $\mathcal{D}(A)$:

invertible operators, one-sided invertible operators, Fredholm operators, simply polar operators, orthogonal projections on Hilbert spaces and algebraic operators.

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The following proposition is shown in [1, Theorem 2.3.2].

1.1. Proposition. *If $A \in \mathcal{R}(X)$ and $B_0 \in \mathcal{G}_2(A)$, then*

$$\mathcal{G}_1(A) = \{B_0 + U - B_0AUAB_0 : U \in \mathcal{L}(X)\}.$$

1.2. Proposition. *If $A \in \mathcal{R}(X)$ and $B \in \mathcal{G}_1(A)$, then*

$$\mathcal{D}(A) = \{U - BAUAB : U \in \mathcal{L}(X)\}.$$

Proof. Let $B_0 = BAB$, then $B_0 \in \mathcal{G}_2(A)$, $B_0A = BABA = BA$ and $AB_0 = ABAB = AB$, hence

$$U - BAUAB = U - B_0AUAB_0 \quad \text{for all } U \in \mathcal{L}(X).$$

Now let $C \in \mathcal{D}(A)$, then $C = B_1 - B_2$ for some $B_1, B_2 \in \mathcal{G}_1(A)$. By Proposition 1.1, there are $U_1, U_2 \in \mathcal{L}(X)$ such that

$$B_j = B_0 + U_j - B_0AU_jAB_0 = B_0 + U_j - BAU_jAB$$

($j = 1, 2$). Hence

$$C = B_1 - B_2 = U - BAUAB$$

with $U = U_1 - U_2$.

Now let $U \in \mathcal{L}(X)$ and define $B_1, B_2 \in \mathcal{L}(X)$ by

$$B_j = B_0 + jU - jB_0AUAB_0 \quad (j = 1, 2).$$

From Proposition 1.1 we know that $B_j \in \mathcal{G}_1(A)$, hence

$$U - BAUAB = U - B_0AUAB_0 = B_1 - B_2 \in \mathcal{D}(A).$$

□

1.3. Proposition. *Let $A \in \mathcal{R}(X)$ and $B \in \mathcal{G}_1(A)$. If $\Psi : \mathcal{L}(X) \rightarrow \mathcal{L}(X)$ is defined by*

$$\Psi(U) = U - BAUAB,$$

then

- (1) $\Psi \in \mathcal{L}(\mathcal{L}(X))$;
- (2) $\Psi^2 = \Psi$;
- (3) $\mathcal{D}(A) = \Psi(\mathcal{L}(X))$.

Proof. (1) is clear; (2) is an easy computation; (3) follows from Proposition 1.2. □

2. Invertible operators

We denote the group of invertible operators in $\mathcal{L}(X)$ by $\mathcal{L}(X)^{-1}$.

2.1. Theorem. *If $A \in \mathcal{R}(X)$, then*

$$A \in \mathcal{L}(X)^{-1} \iff \mathcal{D}(A) = \{0\}.$$

Proof. “ \Rightarrow ”: If $A \in \mathcal{L}(X)^{-1}$, then it is clear that $\mathcal{G}_1(A) = \mathcal{G}_2(A) = \{A^{-1}\}$, hence $\mathcal{D}(A) = \{0\}$.

“ \Leftarrow ”: Take $B \in \mathcal{G}_1(A)$, then, by Proposition 1.2,

$$U = BAUAB \quad \text{for all } U \in \mathcal{L}(X).$$

Therefore $I = BA^2B$, hence $B \in \mathcal{L}(X)^{-1}$,

$$A = ABA^2B = A^2B$$

and

$$A = BA^2BA = BA^2.$$

This gives

$$BA = BA^2B = I \quad \text{and} \quad AB = BA^2B = I,$$

thus $A \in \mathcal{L}(X)^{-1}$. □

3. One-sided invertible operators

We say that $A \in \mathcal{L}(X)$ is *left invertible* [resp. *right invertible*], if there is $B \in \mathcal{L}(X)$ such that

$$BA = I \text{ [resp. } AB = I].$$

It is clear that left or right invertible operators are relatively regular.

3.1. Proposition. *Let $A \in \mathcal{R}(X)$ and $B \in \mathcal{G}_1(A)$.*

- (1) *A is left invertible $\iff BA = I \iff N(A) = \{0\}$.*
- (2) *A is right invertible $\iff AB = I \iff N(B) = \{0\}$.*

Proof. (1) If A is left invertible, then $CA = I$ for some $C \in \mathcal{L}(X)$. It follows that $C \in \mathcal{G}_2(A)$. By Proposition 1.1 there is $U \in \mathcal{L}(X)$ such that

$$B = C + U - CAUAC = C + U - UAC.$$

Hence $BA = CA + UA - UACA = CA = I$ and $N(A) = \{0\}$.

Now assume that $N(A) = \{0\}$. Hence $(I - BA)(X) = \{0\}$. Since $I - BA$ is a projection, we get $BA = I$.
(2) Similar. □

3.2. Theorem. *Suppose that $A \in \mathcal{R}(X)$ and that $A \neq 0$.*

$$A \text{ is right invertible} \iff \mathcal{D}(A) = \{C \in \mathcal{L}(X) : AC = 0\}.$$

Proof. Take $B \in \mathcal{G}_1(A)$.

“ \Rightarrow ”: Let $C \in \mathcal{L}(X)$ and $AC = 0$, then $C = C - BACAB$, hence $C \in \mathcal{D}(A)$, by Proposition 1.2.

Now let $C \in \mathcal{D}(A)$, hence $C = U - BAUAB$ for some $U \in \mathcal{L}(X)$. Since $AB = I$ (Proposition 3.1(2)), we have $C = U - BAU$, therefore $AC = AU - ABAU = AU - AU = 0$.

“ \Leftarrow ”: If $U \in \mathcal{L}(X)$, then $U - BAUAB \in \mathcal{D}(A)$, hence

$$AU = AUAB \quad \text{for all } U \in \mathcal{L}(X).$$

A is right invertible if we have shown that $N(B) = \{0\}$ (Proposition 3.1(2)). To this end assume that $N(B) \neq \{0\}$. Then there is $x_0 \in N(B)$ with $x_0 \neq 0$. By the Hahn-Banach theorem, there is $x^* \in X^*$ (the dual space of X) such that $x^*(x_0) = 1$. We have

$$AUx_0 = AUABx_0 = 0 \quad \text{for all } U \in \mathcal{L}(X).$$

Now let $z \in X$ and let $Ux := x^*(x)z$ ($x \in X$). Then $Ux_0 = z$, thus $Az = 0$. Since $z \in X$ was arbitrary, we get the contradiction $A = 0$. □

3.3. Theorem. *Suppose that $A \in \mathcal{R}(X)$ and that $A \neq 0$.*

$$A \text{ is left invertible} \iff \mathcal{D}(A) = \{C \in \mathcal{L}(X) : CA = 0\}.$$

Proof. Take $B \in \mathcal{G}_1(A)$.

“ \Rightarrow ”: As in the proof of Theorem 3.2.

“ \Leftarrow ”: If $U \in \mathcal{L}(X)$, then $U - BAUA \in \mathcal{D}(A)$, thus

$$UA = BAUA \quad \text{for all } U \in \mathcal{L}(X).$$

Assume that there is $x_0 \in N(A)$ with $x_0 \neq 0$. Let $x^* \in X^*$ and define $U \in \mathcal{L}(X)$ by $Ux = x^*(x)x_0$. Then $UAx = x^*(Ax)x_0$ for all $x \in X$, hence

$$BAUAx = x^*(Ax)BAx_0 = 0 \quad \text{for all } x \in X,$$

thus $BAUA = 0$ and so $UA = 0$. Since $x_0 \neq 0$, we get $x^*(Ax) = 0$ for all $x \in X$ and all $x^* \in X^*$, therefore $A = 0$, a contradiction. Thus we have shown that $N(A) = \{0\}$. Now use Proposition 3.1(1). \square

4. Fredholm operators

An operator $A \in \mathcal{L}(X)$ is called a *Fredholm operator* if $\alpha(A) := \dim N(A) < \infty$ and $\beta(A) := \text{codim } A(X) < \infty$. In this case the *index* $\text{ind}(A)$ of A is defined by

$$\text{ind}(A) = \alpha(A) - \beta(A).$$

We denote the set of all Fredholm operators by $\Phi(X)$. It is well-known (see [4, Theorem 13.2]) that

$$\Phi(X) \subseteq \mathcal{B}(X).$$

Notation.

$$\mathcal{F}(X) = \{F \in \mathcal{L}(X) : \dim F(X) < \infty\};$$

$$\mathcal{K}(X) = \{K \in \mathcal{L}(X) : K \text{ is compact}\}.$$

$\mathcal{F}(X)$ is an ideal in $\mathcal{L}(X)$ and $\mathcal{K}(X)$ is a closed ideal in $\mathcal{L}(X)$. We consider the quotient algebras

$$\mathcal{L}(X)/\mathcal{F}(X) \quad \text{and} \quad \mathcal{L}(X)/\mathcal{K}(X).$$

The elements in $\mathcal{L}(X)/\mathcal{F}(X)$ are denoted by \widehat{A} , hence $\widehat{A} = A + \mathcal{F}(X)$, and the elements in $\mathcal{L}(X)/\mathcal{K}(X)$ are denoted by \widetilde{A} , hence $\widetilde{A} = A + \mathcal{K}(X)$.

4.1. Proposition. *For $A \in \mathcal{L}(X)$ the following assertions are equivalent:*

- (1) $A \in \Phi(X)$;
- (2) \widehat{A} is invertible in $\mathcal{L}(X)/\mathcal{F}(X)$;
- (3) \widetilde{A} is invertible in $\mathcal{L}(X)/\mathcal{K}(X)$.

Proof. [2, Satz 75.11]. \square

4.2. Proposition. *If $A \in \Phi(X)$ and $B \in \mathcal{G}_1(A)$, then $B \in \Phi(X)$ and $\text{ind}(B) = -\text{ind}(A)$.*

Proof. First let $B_0 \in \mathcal{G}_2(A)$, then $AB_0A = A, B_0 = B_0AB_0$,

$$X = \underbrace{(AB_0)(X)}_{=A(X)} \oplus \underbrace{(I - AB_0)(X)}_{=N(B_0)}$$

and

$$X = \underbrace{(B_0A)(X)}_{=B_0(X)} \oplus \underbrace{(I - B_0A)(X)}_{=N(A)}.$$

This shows that $B_0 \in \Phi(X)$ and $\text{ind}(B_0) = -\text{ind}(A)$. Furthermore we have $I - AB_0, I - B_0A \in \mathcal{F}(X)$, thus

$$\widehat{AB_0} = \widehat{I} = \widehat{B_0A}.$$

Proposition 1.1 shows that $B = B_0 + U - B_0AUAB_0$ for some $U \in \mathcal{L}(X)$. Hence $\widehat{B} = \widehat{B_0}$, thus \widehat{B} is invertible in $\mathcal{L}(X)/\mathcal{F}(X)$, therefore $B \in \Phi(X)$.

From $ABA = A$ and the index-theorem ([4, Theorem 13.1]) it follows that

$$\text{ind}(A) = 2\text{ind}(A) + \text{ind}(B),$$

hence $\text{ind}(B) = -\text{ind}(A)$. □

4.3. Corollary. *Suppose that A is relatively regular. Then*

$$A \in \Phi(X) \iff B \in \Phi(X) \quad \text{for all } B \in \mathcal{G}_1(A).$$

Proof. “ \Rightarrow ”: Proposition 4.2.

“ \Leftarrow ”: If $B_0 \in \mathcal{G}_2(A)$, then $B_0 \in \Phi(X)$ and $A \in \mathcal{G}_2(B_0)$. Proposition 4.2 gives $A \in \Phi(X)$. □

4.4. Theorem. *The following assertions are equivalent:*

- (1) $A \in \Phi(X)$;
- (2) $\mathcal{D}(A) \subseteq \mathcal{F}(X)$;
- (3) $\mathcal{D}(A) \subseteq \mathcal{H}(X)$.

Proof. (1) \Rightarrow (2): Take $B_0 \in \mathcal{G}_2(A)$ and $B_1, B_2 \in \mathcal{G}_1(A)$. The proof of Proposition 4.2 shows that $B_1 - B_0, B_2 - B_0 \in \mathcal{F}(X)$, hence $B_1 - B_2 = B_1 - B_0 - (B_2 - B_0) \in \mathcal{F}(X)$.

(2) \Rightarrow (3): Clear.

(3) \Rightarrow (1): Let $B \in \mathcal{G}_1(A)$. Then $\mathcal{D}(A) = \{U - BAUAB : U \in \mathcal{L}(X)\}$. Hence

$$\tilde{U} = \tilde{B}\tilde{A}\tilde{U}\tilde{A}\tilde{B} \quad \text{for all } U \in \mathcal{L}(X).$$

With $U = I$, we get $\tilde{I} = \tilde{B}(\tilde{A})^2\tilde{B}$, hence \tilde{B} is invertible in $\mathcal{L}(X)/\mathcal{H}(X)$. By Proposition 4.1, $B \in \Phi(X)$. Now use Corollary 4.3 to get $A \in \Phi(X)$. □

5. Simply polar operators

In this section we assume that $A \in \mathcal{L}(X)$ is relatively regular, $A \notin \mathcal{L}(X)^{-1}$ and that $A \neq 0$.

We say that A is *simply polar* if $AB = BA$ for some $B \in \mathcal{G}_2(A)$.

5.1. Proposition. *The following assertions are equivalent:*

- (1) A is simply polar;
- (2) $a(A) = d(A) = 1$.

Proof. [2, Satz 72.4 and Satz 101.2], [1, Theorem 5.2] (observe that $A \notin \mathcal{L}(X)^{-1}$ and $A \neq 0$). □

5.2. Theorem. *The following assertions are equivalent:*

- (1) A is simply polar;
- (2) there is $B \in \mathcal{G}_1(A)$ such that $\mathcal{D}(A) = \mathcal{D}(B)$.

Proof. (1) \Rightarrow (2): There is $B \in \mathcal{G}_2(A)$ such that $AB = BA$, hence

$$\underbrace{U - BAUAB}_{\in \mathcal{D}(A)} = \underbrace{U - ABUBA}_{\in \mathcal{D}(B)} \quad \text{for all } U \in \mathcal{L}(X).$$

(2) \Rightarrow (1): From Proposition 1.2 we get that for each $U \in \mathcal{L}(X)$ there is $V \in \mathcal{L}(X)$ such that

$$V - ABVBA = U - BAUAB.$$

Hence there is $U \in \mathcal{L}(X)$ with

$$B - AB^3A = U - BAUAB.$$

It follows that

$$0 = A(U - BAUAB)A = ABA - A^2B^3A^2,$$

hence $A = A^2B^3A^2$. Consequently $N(A^2) = N(A)$ and $A(X) = A^2(X)$, which gives $a(A) = d(A) = 1$ (observe that $A \notin \mathcal{L}(X)^{-1}$). By Proposition 5.1, we derive that A is simply polar. \square

6. Orthogonal projections

In this section X is a complex Hilbert space with the inner product $(\cdot|\cdot)$. A projection $P \in \mathcal{L}(X)$ is called an *orthogonal projection* if $P = P^*$.

6.1. Theorem. *If $P \in \mathcal{L}(X)$ and $P^2 = P$, then*

$$P = P^* \iff \mathcal{D}(P)^* = \mathcal{D}(P),$$

where $\mathcal{D}(P)^* = \{C^* : C \in \mathcal{D}(P)\}$.

Proof. We can assume that $0 \neq P \neq I$. Since $P = P^3$, $P \in \mathcal{G}_2(P)$ and so $\mathcal{D}(P) = \{U - PUP : U \in \mathcal{L}(X)\}$. Hence

$$\mathcal{D}(P)^* = \{U^* - P^*U^*P^* : U \in \mathcal{L}(X)\},$$

and so $\mathcal{D}(P)^* = \mathcal{D}(P)$ if $P = P^*$.

Now assume that $\mathcal{D}(P)^* = \mathcal{D}(P)$. With $U = I$ we have $I - P \in \mathcal{D}(P)$, thus there is $V \in \mathcal{L}(X)$ such that

$$I - P = V - P^*VP^*,$$

thus $0 = P^*(I - P)P^* = P^* - P^*PP^*$, therefore $P^* = P^*PP^*$. Let $Q = P^*P$. Then we have $0 \neq Q$, $Q^* = Q$ and $Q^2 = Q$, hence Q is an orthogonal projection. It follows that $\|Q\| = 1$ ([5, Satz 5.9]). Now let $x \in X$, then

$$\begin{aligned} \|Px\|^2 &= (Px|Px) = (P^*Px) = (Qx|x) \\ &= |(Qx|x)| \leq \|Qx\|\|x\| \leq \|Q\|\|x\|^2 = \|x\|^2, \end{aligned}$$

therefore $\|P\| \leq 1$. Since $0 \neq P = P^2$, we also have $\|P\| \geq 1$. Consequently, $\|P\| = 1$. Use again Satz V.5.9 in [5] to see that $P = P^*$. \square

7. Algebraic operators

An operator $A \in \mathcal{L}(X)$ is called *algebraic* if $p(A) = 0$ for some non-zero polynomial p .

7.1. Proposition. *Let $A \in \mathcal{L}(X)$. Then A is algebraic if and only if $\sigma(A)$ consists of a finite number of poles of $(\lambda I - A)^{-1}$.*

Proof. [4, Chapter V.11]. \square

7.2. Proposition. *Suppose that A is algebraic and simply polar. Then there is a polynomial p such that $p(A) \in \mathcal{G}_2(A)$.*

Proof. [3, Corollary 1.4 and Theorem 2.1]. \square

7.3. Theorem. *Let $A \in \mathcal{R}(X)$, let p be a polynomial and suppose that $p(0) = 0$. Then*

$$A = \pm Ap(A) \iff \mathcal{D}(A) = \{U - p(A)Up(A) : U \in \mathcal{L}(X)\}.$$

Proof. “ \Rightarrow ”: Since $p(0) = 0$, $p(\lambda) = \lambda q(\lambda)$ for a polynomial q , hence $p(A) = Aq(A)$. Let

$$B = \begin{cases} q(A), & \text{if } A = Ap(A) \\ -q(A), & \text{if } A = -Ap(A). \end{cases}$$

Then $ABA = A$, thus $B \in \mathcal{G}_1(A)$ and

$$\begin{aligned} \mathcal{D}(A) &= \{U - BAUAB : U \in \mathcal{L}(X)\} \\ &= \{U - p(A)Up(A) : U \in \mathcal{L}(X)\}. \end{aligned}$$

“ \Leftarrow ”: We have

$$(*) \quad AUA = Ap(A)Up(A)A \quad \text{for all } U \in \mathcal{L}(X).$$

If $B \in \mathcal{G}_1(A)$ it follows that

$$A = ABA = Ap(A)Bp(A)A = p(A)ABAp(A) = Ap(A)^2,$$

thus $A(I - p(A)^2) = 0$. The spectral mapping theorem gives

$$\sigma(A) \subseteq \{0\} \cup \{\lambda \in \mathbb{C} : p(\lambda)^2 = 1\}.$$

Since A is algebraic, each element of $\sigma(A)$ is an eigenvalue of A (see [2, Satz 101.2]). Now take $\lambda \in \sigma(A) \setminus \{0\}$. Then $Ax_0 = \lambda x_0$ for some $x_0 \neq 0$, thus $p(A)x_0 = p(\lambda)x_0 = \pm x_0$.

From $(*)$ we get

$$\begin{aligned} \lambda AUx_0 &= AUAx_0 = Ap(A)Up(A)Ax_0 \\ &= \pm \lambda Ap(A)Ux_0 \end{aligned}$$

for all $U \in \mathcal{L}(X)$. Since $\lambda \neq 0$, we derive

$$AUx_0 = \pm Ap(A)Ux_0 \quad \text{for all } U \in \mathcal{L}(X).$$

Now let $x \in X$. There is $U \in \mathcal{L}(X)$ such that $Ux_0 = x$, hence

$$Ax = \pm Ap(A)x.$$

Therefore $A = \pm Ap(A)$. □

The proof just given shows:

7.4. Corollary. *Let A and p as in Theorem 7.3. Then:*

- (1) $A = Ap(A) \iff \mathcal{D}(A) = \{U - p(A)Up(A) : U \in \mathcal{L}(X)\}$ and $p(\lambda) \neq -1$ for all $\lambda \in \sigma(A)$;
- (2) $A = -Ap(A) \iff \mathcal{D}(A) = \{U - p(A)Up(A) : U \in \mathcal{L}(X)\}$ and $p(\lambda) \neq 1$ for all $\lambda \in \sigma(A)$.

7.5. Examples. *Let $A \in \mathcal{R}(X)$. Then:*

- (1) $A = \pm A^2 \iff \mathcal{D}(A) = \{U - AUA : U \in \mathcal{L}(X)\}$;
- (2) $A^2 = A \iff \mathcal{D}(A) = \{U - AUA : U \in \mathcal{L}(X)\}$ and $-1 \notin \sigma(A)$;
- (3) $A^2 = -A \iff \mathcal{D}(A) = \{U - AUA : U \in \mathcal{L}(X)\}$ and $1 \notin \sigma(A)$;
- (4) $A = \pm A^3 \iff \mathcal{D}(A) = \{U - A^2UA^2 : U \in \mathcal{L}(X)\}$;
- (5) $A = A^3 \iff \mathcal{D}(A) = \{U - A^2UA^2 : U \in \mathcal{L}(X)\}$ and $i, -i \notin \sigma(A)$;
- (6) $A = -A^3 \iff \mathcal{D}(A) = \{U - A^2UA^2 : U \in \mathcal{L}(X)\}$ and $1, -1 \notin \sigma(A)$.

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